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DEPARTMENT OF MATHEMATICS

Subject Code: 18MAT21

Subject: Advanced Calculus & Numerical Methods

Vector Calculus.

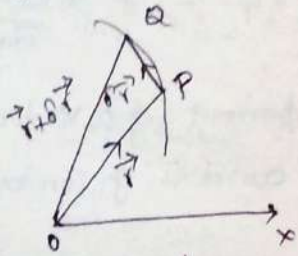
We know that Scalar is a quantity which has only magnitude. and Vector is a quantity having both magnitude and direction. The quantities force, velocity accelaration etc. are vectors.

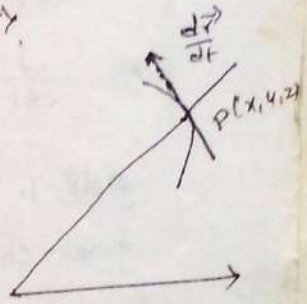
Vector differentiation

Vector function of a single variable and the derivative of a vector.

Let the position vector of a point $P(x,y,z)$ in space be $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ If x,y,z are all functions of a single parameter 't', then $\vec{r}$ is said to be a vector function of 't' which is known as a vector point function usually denoted by $\vec{r} = \vec{r}(t)$. As the parameter 't' varies the point P traces a curve in space. $\therefore \vec{r} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ is called vector Equation of a curve

By giving a small increment δt to t $\vec{r}$ gets an increment $\delta\vec{r}$.





Let P and Q be two neighbouring points on the curve. Let $\overrightarrow{OP} = \vec{r}$ and $\overrightarrow{OQ} = \vec{r} + \delta\vec{r}$.

$\therefore \overrightarrow{PQ} = \delta\vec{r}$

The derivative of the vector $\vec{r}(t)$ is denoted and defined as follows,

$$\text{If } \frac{d\vec{r}}{dt} = \vec{r}'(t) = \lim_{\delta t \to 0} \frac{\vec{r}(t+\delta t) - \vec{r}(t)}{\delta t} \quad \text{--- (1).}$$

Geometrical meaning of the derivative of a vector

As $\delta t \to 0$, $Q \to P$ which means that a chord PQ approaches the tangent to the curve at P. Thus geometrically we can say that $\frac{d\vec{r}}{dt}$ is a vector along the tangent to the curve at P.

Physical meaning of $\frac{d\vec{r}}{dt}$: As $\vec{r} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ represents the position vector of a point moving along a curve

$\vec{v} = \frac{d\vec{r}}{dt}$ gives the velocity of the particle at time 't'.

Further $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$ represents the rate of change of velocity $\vec{v}$ and is called the accelaration of the particle at time 't'.

Further $|\vec{v}| = \left|\frac{d\vec{r}}{dt}\right| = \frac{ds}{dt}$ is the speed of P where s is the arc length measured from a fixed point on the curve on to the point P along the curve.

Note: When a particle moves along the curve velocity vector is directed along the tangent.

2. WKT $\vec{r} = x(t)\,i + y(t)\,j + z(t)\,k$

then $\frac{d\vec{r}}{dt} = \frac{dx}{dt}\,i + \frac{dy}{dt}\,j + \frac{dz}{dt}\,k$

$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\,i + \frac{d^2y}{dt^2}\,j + \frac{d^2z}{dt^2}\,k$$

Properties

1. $\frac{d}{dt}(\vec{F}\cdot\vec{G}) = \vec{F}\cdot\frac{d\vec{G}}{dt} + \vec{G}\cdot\frac{d\vec{F}}{dt}$ where $\vec{F} = \vec{F}(t)$: $\vec{G} = \vec{G}(t)$.

2. $\frac{d}{dt}(\vec{F}\times\vec{G}) = \vec{F}\times\frac{d\vec{G}}{dt} + \frac{d\vec{F}}{dt}\times\vec{G}$

Note 1. Component of a vector $\vec{A}$ along a vector (direction) $\vec{B}$ is the resolved part of $\vec{A}$ and is given by $\vec{A}\cdot\hat{n}$ where $\hat{n} = \frac{\vec{B}}{|\vec{B}|}$.

2. component of a vector $\vec{A}$ $\perp^r$ to the vector $\vec{B}$ (along the normal) is given by,

$|\vec{A} - (\text{resolved part of } \vec{A} \text{ along } \vec{B})| = |\vec{A} - (\vec{A}\cdot\hat{n})\cdot\hat{n}|$

Unit vector Normal to the space curve:

Let A be a fixed point on the curve and let the length of the arc AP be s.

i.e., $\overset{\frown}{AP} = s$ where $P(x,y,z)$ and $\vec{r} = x(t)\,i + y(t)\,j + z(t)\,k$

The unit tangent vector $\hat{T}$ can be put in the form, $\hat{T} = \frac{d\vec{r}}{ds}$

Since $\frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{d\vec{r}/dt}{|d\vec{r}/dt|} = \frac{\vec{T}}{|\vec{T}|}$

we have $\vec{T}\cdot\hat{T} = 1$

Diff wrt 's' we get $2\hat{T}\,\frac{d\hat{T}}{ds} = 0 \Rightarrow \hat{T}\,\frac{d\hat{T}}{ds} = 0 \Rightarrow \hat{T}$ is $\perp$ $\frac{d\hat{T}}{ds}$.

$\Rightarrow \frac{d\hat{T}}{ds}$ is normal to space curve

$\therefore$ unit normal vector $\hat{N} = \frac{d\hat{T}}{ds} \Big/ \left|\frac{d\hat{T}}{ds}\right|$

$\hat{N}$ is also called the principal vector normal.

1. If $x = t^2+1 : y = 4t-3 : z = 2t^2-6t$ represents the parametric Equation of a curve determine (a) The unit tangent vector at any point on the Curve (b) The angle between the tangents at $t=1$ and $t=2$

Sol We have the vector Equation of curve,

$$\vec{r} = x(t)\,i + y(t)\,j + z(t)\,K$$
$$= (t^2+1)\,i + (4t-3)\,j + (2t^2-6t)\,K$$

$\therefore$ Tangent vector, $\vec{T} = \frac{d\vec{r}}{dt} = 2t\,i + 4j + (4t-6)\,K$

a) unit tangent vector $\hat{T} = \frac{\vec{T}}{|\vec{T}|} = \frac{2t\,i + 4j + (4t-6)K}{\sqrt{(2t)^2 + 4^2 + (4t-6)^2}}$

$$= \frac{t\,i + 2j + (2t-3)K}{\sqrt{5t^2-12t+13}}$$

(b) $\vec{T}_{t=1} = 2i + 4j - 2K$ $\quad \vec{T}_{t=2} = 4i + 4j + 2K$

Then $\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|}$ where θ is the angle between $\vec{A}$ & $\vec{B}$

$$= \frac{2\cdot4 + 4\cdot4 - 2\cdot2}{\sqrt{4+16+4}\,\sqrt{16+16+4}} = \frac{20}{\sqrt{24}\cdot6} = \frac{20}{12\sqrt{6}} = \frac{5}{3\sqrt{6}}$$

$\therefore \theta = \cos^{-1}\frac{5}{3\sqrt{6}}$ is the required angle.

2. Find angle between the tangents to the curve

$\vec{r} = \left(t - \frac{t^3}{3}\right)i + t^2 j + \left(t + \frac{t^3}{3}\right)K$ at $t = \pm 3$.

Sol: $\vec{T} = \frac{d\vec{r}}{dt} = (1-t^2)\,i + 2t\,j + (1+t^2)\,K$. is the tangent vector

$(\vec{T})_{t=3} = -8i + 6j + 10K$: $\vec{T}_{t=-3} = -8i - 6j + 10K$

$\therefore \cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|} = \frac{128}{200} = \frac{16}{25}$

$\therefore \theta = \cos^{-1}\frac{16}{25}$

3. Find the unit normal vector to the curve $\vec{r}(t) = 4\sin t\,\hat{i} + 4\cos t\,\hat{j}$

Sol: Wkt if $\hat{T}$ is the unit tangent vector then unit normal vector is given by $\hat{N} = \frac{d\hat{T}}{ds} \Big/ \left|\frac{d\hat{T}}{ds}\right|$

$\vec{T} = \frac{d\vec{r}}{dt} = 4\cos t\,\hat{i} - 4\sin t\,\hat{j} + 3\hat{k}$

$\hat{T} = \frac{\vec{T}}{|\vec{T}|} = \frac{4\cos t\,\hat{i} - 4\sin t\,\hat{j} + 3\hat{k}}{5}$ is the unit tgt vector.

$\frac{d\hat{T}}{ds} = \frac{d\hat{T}}{dt} \Big/ \frac{ds}{dt} = \frac{d\hat{T}}{dt} \Big/ \left|\frac{d\vec{r}}{dt}\right| = \frac{\frac{1}{5}[-4\sin t\,\hat{i} - 4\cos t\,\hat{j}]}{5} = \frac{-4}{25}(\sin t\,\hat{i} + \cos t\,\hat{j})$

$\Rightarrow \left|\frac{d\hat{T}}{ds}\right| = \frac{4}{25}$

$\therefore \hat{N} = -(\sin t\,\hat{i} + \cos t\,\hat{j})$, is the required unit normal vector.

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4. Find the unit tangent vector and the unit normal vector to the curve $\vec{r} = \cos 2t\,\hat{i} + \sin 2t\,\hat{j} + t\,\hat{k}$ at $x = 1/\sqrt{2}$.

Sol: The unit tangent vector, $\hat{T} = \frac{d\vec{r}}{dt} \Big/ \left|\frac{d\vec{r}}{dt}\right|$

and unit normal vector $\hat{N} = \frac{d\hat{T}}{ds} \Big/ \left|\frac{d\hat{T}}{ds}\right|$

$\vec{r} = \cos 2t\,\hat{i} + \sin 2t\,\hat{j} + t\,\hat{k}$

$\frac{d\vec{r}}{dt} = -2\sin 2t\,\hat{i} + 2\cos 2t\,\hat{j} + \hat{k} \qquad \left|\frac{d\vec{r}}{dt}\right| = \sqrt{5}$

$\therefore \hat{T} = \frac{-2\sin 2t\,\hat{i} + 2\cos 2t\,\hat{j} + \hat{k}}{\sqrt{5}}$

$\frac{d\hat{T}}{ds} = \frac{d\hat{T}}{dt} \Big/ \frac{ds}{dt} = \frac{d\hat{T}}{dt} \Big/ \left|\frac{d\vec{r}}{dt}\right| = \frac{-4\cos 2t\,\hat{i} - 4\sin 2t\,\hat{j}}{\sqrt{5}} \Big/ \sqrt{5} = \frac{-4}{5}[\cos 2t\,\hat{i} + \sin 2t\,\hat{j}]$

$\left|\frac{d\hat{T}}{ds}\right| = \frac{4}{5}(1) = \frac{4}{5}$

$\hat{N} = -[\cos 2t\,\hat{i} + \sin 2t\,\hat{j}]$

$x = \frac{1}{\sqrt{2}} \therefore \cos 2t = \frac{1}{\sqrt{2}} = \cos 45^\circ \Rightarrow 2t = \frac{\pi}{4} \Rightarrow t = \frac{\pi}{8}$

$\therefore \hat{T} = \frac{-2\sin\frac{\pi}{4}\,\hat{i} + 2\cos\frac{\pi}{4} + \hat{k}}{\sqrt{5}} = \frac{-\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j} + \hat{k}}{\sqrt{5}}$ and $\hat{N} = \frac{-1}{\sqrt{2}}(\hat{i} + \hat{j})$.

5. A particle moves along a curve whose parametric Equations are $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$ where t is the time. Find velocity and accelaration at any time t and also their magnitudes at $t=0$ (initial vel & acce

Sol We have $\vec{r} = e^{-t}\,i + 2\cos 3t\,j + 2\sin 3t\,k$

$\vec{v} = \frac{d\vec{r}}{dt} = -e^{-t}\,i - 6\sin 3t\,j + 6\cos 3t\,k$

$\vec{a} = \frac{d^2\vec{r}}{dt^2} = e^{-t}\,i - 18\cos 3t\,j - 18\sin 3t\,k$

$|\vec{v}| = \sqrt{e^{-2t} + 36}$ when $t=0$ $\quad |\vec{v}| = \sqrt{37}$

$|\vec{a}| = \sqrt{e^{-2t} + 324}$ when $t=0$ $\quad |\vec{a}| = \sqrt{325}$.

6. A particle moves along the curve $x = 1-t^3$; $y = 1+t^2$ and $z = 2t-5$

a) Determine its velocity and accelaration.

b) Find components of velocity and accelaration at $t=1$ in the direction $2i+j+2k$

c) $\vec{r} = (1-t^3)\,i + (1+t^2)\,j + (2t-5)\,k$ is the position vector of the particle.

Sol $\vec{r} = (1-t^3)\,i + (1+t^2)\,j + (2t-5)\,k$ is the position vector of the particle.

$\vec{v} = -3t^2\,i + 2t\,j + 2k$; $\vec{a} = -6t\,i + 2j$.

$(\vec{v})_{t=1} = -3i + 2j + 2k$ $\quad (\vec{a})_{t=1} = -6i + 2j$

component of velocity in the direction of $2i+j+2k = \vec{v}\cdot\hat{A}$

$= (-3i + 2j + 2k)\cdot\left(\frac{2i+j+2k}{3}\right) = \frac{-6+2+4}{3} = 0$

comp of accelaration in the direction of $2i+j+2k = \vec{a}\cdot\hat{A}$

$= (-6i + 2j)\cdot\frac{2i+j+2k}{3} = \frac{-12+2}{3} = \frac{-10}{3}$

7. Find the components of velocity and accelaration at $t=2$ on the curve $\vec{r} = (t^2+1)\,i + (4t-3)\,j + (2t^2-6t)\,k$ in the direction $i+2j+2k$

Ans: $\vec{v} = 4i + 4j + 2k$ $\quad \vec{a} = 2i + 4k$ $\quad \frac{16}{3}, \frac{10}{3}$ comp.

8. A particle moves along the curve C $x = t^3 - 4t$; $y = t^2 + 4t$; $z = 8t^2 - 3t^3$ where t denotes time. Find the components of its accelaration at $t=2$ along the tangent and normal.

Ans: $\vec{v} = 8i + 8j - 4k$; $\vec{a} = 12i + 2j - 20k$ $\quad 16, 2\sqrt{73}$.

9. The position vector of a moving particle at time t is $\vec{r} = t^2 i - t^3 j$. Find the tangential and normal components of its acceleration at $t=1$.

Sol: $V = \frac{d\vec{r}}{dt} = 2t\,i - 3t^2 j + 4t^3 K$.

$\vec{a} = \frac{d^2\vec{r}}{dt^2} = 2i - 6t\,j + 12t^2 K$

$\therefore (\vec{v})_{t=1} = 2i - 3j + 4K =$ $(\vec{a})_{t=1} = 2i - 6j + 12K. = \vec{A}$

Since velocity is along the tangent to the curve the component of acceleration along the tgt is given by $\vec{A}.\hat{n} =$

$= (2i - 6j + 12K) \cdot \frac{2i - 3j + 4k}{\sqrt{4+9+16}} = \frac{4+18+48}{\sqrt{29}} = \frac{70}{\sqrt{29}}$.

component of acceleration along the Normal is $|\vec{A}$ - resolved part of A along tgt$|$

$= \left| (2i - 6j + 12K) - \frac{70}{\sqrt{29}} \left(\frac{2i - 3j + 4K}{\sqrt{29}} \right) \right| = \left| \frac{-82i + 36j + 68K}{29} \right| = \sqrt{\frac{436}{29}}$

10. The P.V. of a particle at time t is $\vec{r} = \cos(t-1)\,i + \sinh(t-1)\,j + at^3 k$. Find the value of a such that acceleration at time $t=1$ is normal to the P.V.

Soln: $\vec{v} = \frac{d\vec{r}}{dt} = -\sin(t-1)\,i + \cosh(t-1)\,j + 3at^2 K$.

$\vec{a} = \frac{d^2\vec{r}}{dt^2} = -\cos(t-1)\,i + \sinh(t-1)\,j + 6at\,k$

$(\vec{a})_{t=1} = -i + 6aK = \vec{A}$ $(\vec{r})_{t=1} = i + aK = \vec{R}$

We need to find a such that $\vec{A}.\vec{R} = 0$

$\Rightarrow -1 + 6a^2 = 0 \Rightarrow 6a^2 = 1 \Rightarrow a = \pm 1/\sqrt{6}$

11. A particle moves so that its position vector is given by $\vec{r} = \cos\omega t\,i + \sin\omega t\,j$. where ω is a constant. Show that (a) velocity $\vec{v}$ is $\perp^r$ to $\vec{r}$ (b) $\vec{r} \times \vec{v}$ is a constant vector (c) acceleration $\vec{a}$ is directed towards the origin and has magnitude proportional to distance from the origin.

Sol:

$\vec{r} = \cos\omega t\, i + \sin\omega t\, j$

$\vec{v} = \frac{d\vec{r}}{dt} = -\omega \sin\omega t\, i + \omega \cos\omega t\, j$

$\vec{a} = \frac{d^2\vec{r}}{dt^2} = -\omega^2 \cos\omega t\, i - \omega^2 \sin\omega t\, j$

Now $\vec{v}.\vec{r} = 0 \Rightarrow \vec{v} \perp \vec{r}$.

$\vec{r} \times \vec{v} = \omega k$ is a constant vector

$\vec{a} = -\omega^2 \vec{r}$

$|\vec{a}| = |-\omega^2 \vec{r}| = \omega^2 . |\vec{r}| = \omega^2 \cdot 1$ $\quad \vec{r} = \vec{OP}$ then $-\vec{r} = \vec{PO}$

$\therefore \vec{a} = \omega^2 \vec{PO}$, where O is origin P is any point on the curve. This shows that the acceleration is directed towards the origin.

Also $|\vec{a}| = \omega^2 |\vec{r}| \Rightarrow |\vec{a}| \propto |\vec{r}|$ $\quad |\vec{r}|$ is distance of P from O.

Thus the magnitude of the acceleration is proportional to the distance from origin.

Find tangential and normal components of acceleration of a particle moving along curve $x(t) = t^2$, $y(t) = -t^3$, $z(t) = t^4$ at $t = 1$.

A particle moves on the curves $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$, where t is time. Find the components of velocity and acceleration at time $t = 1$ in the direction of $i - 3j + 2k$

tangential component of acc^n = comp of acc^n along the tangent

= comp of acc^n along the velocity) in the direction of velocity.

Normal comp of acc^n = comp of acc^n along the normal

= " $\perp$ to tgt

= comp of acc^n in dir $\perp$ to velocity.

Scalar and Vector Fields:

If to Every point (x,y,z) of a region R in space there corresponds a scalar $\phi(x,y,z)$ then ϕ is called a scalar point function and we say that a scalar field ϕ has been defined in R.

If to every point (x,y,z) of a region R in space corresponds a vector $\vec{A}(x,y,z)$ then $\vec{A}$ is called a vector point function and we say that a vector field $\vec{A}$ has been defined in R.

Vector differential operator ∇ (Del or Nabla) is defined as,

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k = \sum \frac{\partial}{\partial x} i$$

The Laplacian operator ∇^2 is defined as, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Gradient of a Scalar Field:

If $\phi = \phi(x,y,z)$ is a continuously differentiable scalar function then The gradient of ϕ grad ϕ or $\nabla\phi$ is defined as,

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$$

$\nabla\phi$ is a vector quantity.

Divergence of a vector field:

If $\vec{A}(x,y,z)$ is a continuously differentiable vector function then divergence of $\vec{A}$ $\text{div}\,\vec{A}$ or $\nabla \cdot \vec{A}$ is a scalar defined by,

$$\vec{A} = a_1 i + a_2 j + a_3 k$$

$$\nabla \cdot \vec{A} = \text{div}\,\vec{A} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}.$$

Curl of a vector field:

If $\vec{A}(x,y,z) = a_1 i + a_2 j + a_3 k$ is a continuously diff. vector function then curl of $\vec{A}$ is defined to be $\nabla \times \vec{A}$ or curl

$$\nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

Laplacian: If $\phi = \phi(x,y,z)$ is continuously diff. scalar function and $\vec{A} = a_1 i + a_2 j + a_3 k$ is continuously diff. vector function then Laplacian of ϕ as well as $\vec{A}$ is defined as.

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$\nabla^2 \vec{A} = \frac{\partial^2 \vec{A}}{\partial x^2} + \frac{\partial^2 \vec{A}}{\partial y^2} + \frac{\partial^2 \vec{A}}{\partial z^2}$$

Note: If ϕ is a scalar function then the Equation $\nabla^2 \phi = 0$ is called Laplace's Equation and a function which satisfies Laplace Equation is called a harmonic function.

Laplacian of a scalar function is a scalar quantity and Laplacian of a vector function is a vector quantity.

If ϕ is a scalar function then $\text{grad}\,\phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$

$\therefore \text{div}(\text{grad}\,\phi) = \nabla \cdot \nabla \phi$

$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \cdot \left(\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$\therefore \nabla \cdot (\nabla \phi) = \nabla^2 \phi.$$

Geometrical Meaning of the gradient.

If $\phi = \phi(x,y,z)$ is a scalar function then grad ϕ is a vector normal to the surface $\phi(x,y,z) = C$

let $\vec{r}$ be the p.v. of any point $P(x,y,z)$ on the surface $\phi(x,y,z) = C$

i.e $\vec{r} = xi + yj + zk$

$\therefore \frac{d\vec{r}}{dt} = \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k$ is tangential to the surface at P.

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

consider $\nabla\phi \cdot \frac{d\vec{r}}{dt} = \frac{\partial\phi}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\cdot\frac{dz}{dt}$

Now Consider $\phi = \phi(x,y,z)$ with $x = x(t)$: $y = y(t)$: $z = z(t)$ c is const

Diff. w.r.t.t we get $\frac{d\phi}{dt} = 0 \Rightarrow \frac{\partial\phi}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\cdot\frac{dz}{dt} = 0$

$\Rightarrow \nabla\phi \cdot \frac{d\vec{r}}{dt} = 0 \Rightarrow \nabla\phi \perp \frac{d\vec{r}}{dt}$.

Since $\frac{d\vec{r}}{dt}$ is a vector tangential to the surface

Hence $\nabla\phi$ is along the normal to the surface $\phi(x,y,z) = C$ at P.

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1. As $\nabla\phi$ is a vector normal to the surface $\phi = C$ the unit normal vector to the surface is $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

2. The angle between two surfaces is defined as the angle between their normals. If $\phi_1(x,y,z) = c_1$ and $\phi_2(x,y,z) = c_2$ are two surfaces then

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}$$ where θ is the angle between the normals.

If $\theta = \pi/2$ then the surfaces are said to intersect each other orthogonally.

$\cos\theta = 0 \Rightarrow \nabla\phi_1 \cdot \nabla\phi_2 = 0$.

This is the condition for the surfaces to intersect at right angles.

If $\phi = \phi(x,y,z)$ is a scalar function and $\vec{a}$ is a given function then $\nabla\phi \cdot \hat{a}$ where $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$ is called the directional derivative of ϕ along $\vec{a}$.

The directional derivative is maximum along $\nabla\phi$ and its maximum value is $|\nabla\phi|$.

If $\phi = \phi(x,y,z) = C$ is the equation of a surface and $P(x_1, y_1, z_1)$ is a point on it then the equation of the tangent plane at P is,

$$A(x-x_1) + B(y-y_1) + C(z-z_1) = 0$$

The Equation of Normal line is $\frac{x-x_1}{A} = \frac{y-y_1}{B} = \frac{z-z_1}{C}$

where $A = \left(\frac{\partial\phi}{\partial x}\right)_{(x_1,y_1,z_1)}$: $B = \left(\frac{\partial\phi}{\partial y}\right)_{(x_1,y_1,z_1)}$: $C = \left(\frac{\partial\phi}{\partial z}\right)_{(x_1,y_1,z_1)}$

If $\vec{V} = \vec{V}(x,y,z)$ represents any physical quantity, the divergence of $\vec{V}$ gives the rate at which the physical quantity is originating at that point per unit volume.

flows out

A vector $\vec{V}$ whose divergence is zero is called a Solenoidal vector.

Curl means rotation. A vector function $\vec{V} = \vec{V}(x,y,z)$ is said to be irrotational if curl $\vec{V} = 0$

1. If $\phi = x^2y + y^2z + z^2x$ find $\nabla\phi$ and $\nabla^2\phi$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = i(2xy+z^2) + j(x^2+2yz) + k(y^2+2xz)$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 2(x+y+z).$$

2. If $\vec{A} = x^2yz\,i + y^2zx\,j + z^2xy\,k$ find div $\vec{A}$, curl $\vec{A}$ and $\nabla^2\vec{A}$.

$$\text{div}\,\vec{A} = 6xyz$$

$$\text{curl}\,\vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & y^2zx & z^2xy \end{vmatrix} = x(z^2-y^2)\,i + y(x^2-z^2)\,j + z(y^2-x^2)\,k.$$

$$\nabla^2\vec{A} = 2yz\,i + 2xz\,j + 2xy\,k.$$

3. Given $\vec{r} = xi + yj + zk$, $\phi = xyz$ and $\vec{A} = x^2 i + y^2 j + z^2 k$ find $(\vec{r}\cdot\nabla)\phi$, $(\vec{r}\cdot\nabla)\vec{A}$. $\vec{r}\cdot\nabla\phi$,

$\phi = x^2y^2z^3$
$\vec{r} = 2x\,i + 3y\,j + 4z\,k$
$\vec{r}\cdot\nabla\phi$, $\vec{r}\times\nabla\phi$, $\nabla|\vec{r}|^2$

$$(\vec{r}\cdot\nabla)\phi = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)(xyz)$$

$$= xyz + yxz + zxy = 3xyz$$

$$(\vec{r}\cdot\nabla)\vec{A} = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)(x^2 i + y^2 j + z^2 k)$$

$$= x\cdot 2x\,i + y\,2y\,j + z\cdot 2z\,k = 2x^2 i + 2y^2 j + 2z^2 k$$

4. Find the unit normal vector to the surface $x^2y + 2xz = 4$ at $(2,-2,3)$.

Sol. $\phi = x^2y + 2xz - 4$

$$\nabla\phi = (2xy + 2z)\,i + x^2 j + 2x\,k$$

$$(\nabla\phi)_{(2,-2,3)} = -2i + 4j + 4k$$

$$\therefore \text{unit normal vec. } \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{+2(-i+2j+2k)}{\sqrt{(1+4+4)2^2}} = \frac{2(-i+2j+2k)}{2\cdot 3}$$

i) $xy^3z^2 = 4$ at $(-1,-1,2)$ ⑤ $4z + 3x + xy = c$ at $P(-1,2,3)$

ii) $x^2y - 2xz + 2y^2z^4 = 10$ at $(2,1,-1)$.

5. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ along $2i - j - 2k$.

Sol. $\phi = x^2yz + 4xz^2$

$$\nabla\phi = (2xyz + 4z^2)\,i + x^2z\,j + (x^2y + 8xz)\,k$$

$$(\nabla\phi)_{(1,-2,-1)} = 8i - j - 10k \qquad \hat{n} = \frac{2i-j-2k}{3} \qquad \therefore \text{D.D.} = \nabla\phi\cdot\hat{n} = \frac{37}{3}.$$ ④

6. i. $\phi = 4xz^3 - 3x^2y^2z$ at $(2,-1,2)$ along $2i - 3j + 6k$ ⑥ $\phi = 2e^{2x-y+z}$ at $P(1,3,1)$ in the dir towards $Q(2,1,3)$

ii) $\phi = \frac{xz}{x^2+y^2}$ at $(1,-1,1)$ in the dir. of $\vec{A} = i - 2j + k$. $-1 \le t \le 1$

iii. $f(x,y,z) = x^2y^2z^2$ at $(1,1,-1)$ in the dir. of tgt to the curve $x = e^t$, $y = 1 + 2\sin t$, $z = t - \cos t$, where

6. Find D.D of the ϕ xyz along the direction of the normal to the surface $xy^2 + yz^2 + zx^2 = 3$ at $(1,1,1)$. Also find the eqn of tgt plane & normal line to the surface

7. In which direction the D.D. of x^2y^3 is max. at $(2,1,-1)$ & find the magnitude of this max.

Module - 2: Differential equations

* An equation which contains the dependent variable, independent variable and its derivatives is called a differential equation.

* The highest derivative appearing in the equation is called the order and the power of the highest order derivative is called the degree of a differential equation

Solution of second order homogenous differential equations:

Standard form: $a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0$

Put $D = \frac{d}{dx}$

$$(a_0 D^2 + a_1 D + a_2) y = 0$$

Step-1: Write the auxiliary equation by replacing

D by m.

i.e., $a_0 m^2 + a_1 m + a_2 = 0$

Step-2: Solve the auxiliary equation.

Case-i,: If the roots are real and distinct (say m_1, m_2) then:

$$\boxed{y = C_1 e^{m_1 x} + C_2 e^{m_2 x}}$$

Case-ii,: If the roots are repeated, then:

$$\boxed{y = (C_1 + C_2 x) e^{mx}}$$

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4. Case-iii;: If the roots are complex (say $\alpha \pm i\beta$), then:

$$\boxed{y = e^{\alpha x}\left(C_1 \cos\beta x + C_2 \sin\beta x\right)}$$

1. Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

Sol: Given, $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

Put $D = \frac{dy}{dx}$

$$(D^2 - 5D + 6)y = 0$$

The auxiliary equation is:

$$m^2 - 5m + 6 = 0$$

$$m^2 - 2m - 3m + 6 = 0$$

$$m(m-2) - 3(m-2) = 0$$

$$\boxed{m = 2,\ m = 3}$$

The general solution is:

$$\boxed{y = C_1 e^{2x} + C_2 e^{3x}}$$

2. Solve $y'' - 6y' + 9y = 0$

Sol: Given, $y'' - 6y' + 9y = 0$

$$(D^2 - 6D + 9)y = 0$$

The auxiliary equation is:

$$m^2 - 6m + 9 = 0$$

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$m^2-3m-3m+9=0$

$m(m-3)-3(m-3)=0$

$\boxed{m=3,3}$

∴ The general solution is:

$$\boxed{y=(C_1+C_2x)e^{3x}}$$

3: Solve $y''+y'+y=0$

Sol: Given, $y''+y'+y=0$

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0$$

Put $D=\frac{dy}{dx}$

$$(D^2+D+1)y=0$$

The auxiliary equation is:

$$(m^2+m+1)=0$$

$$m=\frac{-1\pm\sqrt{1-4\cdot1\cdot1}}{2}$$

$$m=\frac{-1\pm\sqrt{-3}}{2}$$

$$\boxed{m=\frac{-1}{2}\pm\frac{\sqrt{3}}{2}i}$$

$\alpha=\frac{-1}{2}$, $\beta=\frac{\sqrt{3}}{2}$

∴ The general solution is:

$$\boxed{y=e^{\frac{-1}{2}x}\left(C_1\cos\frac{\sqrt{3}}{2}x+C_2\sin\frac{\sqrt{3}}{2}x\right)}$$

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4. Solve: $y'' - 10y' + 25y = 0$

Sol: Given, $y'' - 10y' + 25y = 0$

$$(D^2 - 10D + 25)y = 0$$

The auxiliary equation is:

$$m^2 - 10m + 25 = 0$$

$$m^2 - 5m - 5m + 25 = 0$$

$$m(m-5) - 5(m-5) = 0$$

$$\boxed{m = 5, 5}$$

∴ The general solution is: $\boxed{y = (C_1 + C_2 x)e^{5x}}$

5. Solve: $(D^2 + 2D + 10)y = 0$

Sol: Given, $(D^2 + 2D + 10)y = 0$

The auxiliary equation is:

$$m^2 + 2m + 10 = 0$$

$$m = \frac{-2 \pm \sqrt{4 - 4 \cdot 1 \cdot 10}}{2}$$

$$m = \frac{-2 \pm \sqrt{4 - 40}}{2}$$

$$m = \frac{-2 \pm \sqrt{-36}}{2}$$

$$m = \frac{-2 \pm 6i}{2} \Rightarrow \boxed{m = -1 \pm 3i}$$

$\alpha = -1$

$\beta = 3$

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∴ The general solution is: $\boxed{y = e^{-1x}(C_1 \cos 3x + C_2 \sin 3x)}$

6. Solve $\frac{d^2y}{dx^2} + y = 0$

Sol: Given, $\frac{d^2y}{dx^2} + y = 0$

Put $D = \frac{dy}{dx}$

$D^2y + y = 0$

$(D^2 + 1)y = 0$

The auxiliary equation is: $m^2 + 1 = 0$

$m^2 = -1$

$m = \pm\sqrt{-1}$

$\boxed{m = \pm i}$ $\quad \alpha = 0$, $\beta = 1$

∴ The general solution is: $y = e^{0x}(C_1 \cos x + C_2 \sin x)$

$\boxed{y = (C_1 \cos x + C_2 \sin x)}$

7. Solve: $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} - 4x = 0$

Sol: Given, $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} - 4x = 0$

Put $D = \frac{dx}{dt}$

$(D^2 - 3D - 4)x = 0$

The auxiliary equation is:

$m^2 - 3m - 4 = 0$

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$m^2 - 4m + 1m - 4 = 0$

$m(m-4) + 1(m-4) = 0$

$\boxed{m = 4, m = -1}$

∴ The general solution is: $\boxed{x = C_1 e^{4t} + C_2 e^{-t}}$

8. Solve: $\frac{d^2y}{dx^2} + 8y = 0$ given that $y(0) = 1$ and $y'(0) = 2\sqrt{2}$

Sol: Given, $\frac{d^2y}{dx^2} + 8y = 0$

Put $D = \frac{dy}{dx}$

$(D^2 + 8)y = 0$

Auxiliary equation is: $m^2 + 8 = 0$

$m^2 = -8$

$m = \pm\sqrt{-8}$

$\boxed{m = \pm 2\sqrt{2}\, i}$

The general solution is: $\boxed{y = C_1 \cos 2\sqrt{2}x + C_2 \sin 2\sqrt{2}x}$

When $x = 0, y = 1$

$\boxed{1 = C_1}$

Differentiating wrt x: $y' = -C_1 \sin 2\sqrt{2}x \cdot 2\sqrt{2} + C_2 \cos 2\sqrt{2}x \cdot 2\sqrt{2}$

When $x = 0$, $y' = 2\sqrt{2}$

$2\sqrt{2} = C_2\, 2\sqrt{2}$

$\boxed{C_2 = 1}$

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∴ Solution is: $y = \cos 2\sqrt{2}x + \sin 2\sqrt{2}x$

1. Solve $y'' + y' - 2y = 0$ given that $y=0$ and $y'=3$ when $x=0$.

Sol: Given, $y'' + y' - 2y = 0$

Put $D = \frac{dy}{dx}$

$$(D^2 + D - 2)y = 0$$

The auxiliary equation is:

$$m^2 + m - 2 = 0$$

$$m^2 + 2m - 1m - 2 = 0$$

$$m(m+2) - 1(m+2) = 0$$

$$m = 1, m = -2$$

∴ The general solution is: $y = C_1 e^{x} + C_2 e^{-2x}$

When $x=0$, $y=0$

$$0 = C_1 + C_2 \longrightarrow (1)$$

$$y' = C_1 e^{x} + C_2 e^{-2x}(-2)$$

$$3 = C_1 + C_2(-2)$$

$$3 = C_1 - 2C_2 \longrightarrow (2)$$

(1) − (2) ⇒

$$-3 = 3C_2 \Rightarrow C_2 = -1$$

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From ① $C_1 + C_2 = 0$

$C_1 - 1 = 0$

$\boxed{C_1 = 1}$

∴ The particular solution is: $\boxed{y = e^x - e^{-2x}}$

10. Solve $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

Sol: Given $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

Put $D = \frac{dy}{dx}$

$$(D^3 - 6D^2 + 11D - 6)\, y = 0$$

The auxiliary equation is:

$$m^3 - 6m^2 + 11m - 6 = 0$$

m=0, m=1, m=-1, m=2, m=3

$\boxed{m=1}$ is a root

1	1	−6	11	−6
	(+)0	(+)1	(+)−5	(+)6
	1	−5	6	0

$m^2 - 5m + 6 = 0$

$m^2 - 2m - 3m + 6 = 0$ ∴

$m(m-2) - 3(m-2) = 0$

$\boxed{m = 2,\ m = 3}$

∴ $m = 1, 2, 3$

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∴ The general solution is: $\boxed{y = C_1 e^{x} + C_2 e^{2x} + C_3 e^{3x}}$

11. Solve: $\frac{d^3y}{dx^3} + y = 0$

Sol: Given, $\frac{d^3y}{dx^3} + y = 0$

Put $D = \frac{dy}{dx}$

$$(D^3 + 1)y = 0$$

The auxiliary equation is: $m^3 + 1 = 0$

$\boxed{m = -1}$ is a root

-1	1	0	0	1
	$+0$	-1	1	-1
	1	-1	1	0

$\therefore m^2 - m + 1 = 0$

$$m = \frac{1 \pm \sqrt{1 - 4 \cdot 1 \cdot 1}}{2}$$

$$m = \frac{1 \pm \sqrt{-3}}{2}$$

$$m = \frac{1}{2} \pm \frac{\sqrt{3}\, i}{2}$$

$\therefore \boxed{m = -1, \frac{1}{2} \pm \frac{\sqrt{3}}{2} i}$

∴ The general solution is: $\boxed{y = C_1 e^{-x} + e^{\frac{x}{2}}\left(C_2 \cos\frac{\sqrt{3}}{2}x + C_3 \sin\frac{\sqrt{3}}{2}x\right)}$

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2. Solve: $(D^3 + D^2 + D + 1)y = 0$

Sol: Given, $(D^3 + D^2 + D + 1)y = 0$

The auxiliary equation is: $m^3 + m^2 + m + 1 = 0$

$\boxed{m = -1}$ is a root.

	1	1	1	1
-1	0	-1	0	-1
	1	0	1	0

$\therefore m^2 + 1 = 0$

$m^2 = -1$

$m = \pm\sqrt{-1}$

$\boxed{m = \pm i}$ $\quad \alpha = 0$, $\beta = 1$

$\therefore \boxed{m = -1, m = \pm i}$

$\therefore$ The general solution is: $\boxed{y = C_1 e^{-x} + C_2 \cos x + C_3 \sin x}$

13. Solve $(D^4 + 4D^3 - 5D^2 - 36D - 36)y = 0$

Sol: Given, $(D^4 + 4D^3 - 5D^2 - 36D - 36)y = 0$

The auxiliary equation is: $m^4 + 4m^3 - 5m^2 - 36m - 36 = 0$

$\boxed{m = 2}$ is a root.

	1	4	-5	-36	-36
-2	0	-2	-4	$+18$	$+36$
	1	2	-9	-18	0

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$m^3 + 2m^2 - 9m - 18 = 0$

$\boxed{m = -2}$ is a root

-2	1	2	-9	-18
	0	-2	0	$+18$
	1	0	-9	0

$m^2 - 9 = 0$

$m^2 = 9$

$\boxed{m = \pm\sqrt{3}.\sqrt{3}}$

$\therefore$ ~~$\boxed{m = -2, -2, +\sqrt{3}, -\sqrt{3}}$~~

$\boxed{m = -2, -2, 3, -3}$

$\therefore$ The general solution is : $\boxed{y = (c_1 + c_2 x)e^{-2x} + c_3 e^{3x} + c_4 e^{-3x}}$

14. If $\frac{d^4x}{dt^4} = m^4 x$. Show that $\boxed{x = c_1 \cos mt + c_2 \sin mt + c_3 \cosh mt + c_4 \sinh mt}$

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Sol: Given, $\frac{d^4x}{dt^4} = m^4 x$

$D^4 x = m^4 x$

$(D^4 - m^4)x = 0$

The auxiliary equation is :

$(p^4 - m^4)x = 0$

$(p^2)^2 - (m^2)^2 = 0$

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$(p^2+m^2)(p^2-m^2)=0$

$p^2+m^2=0$ $\qquad$ $p^2-m^2=0$

$p^2=-m^2$ $\qquad$ $p^2=m^2$

$\boxed{p=\pm im}$ $\qquad$ $\boxed{p=\pm m}$

∴ The general solution is: $x=C_1\cos mt+C_2\sin mt+C_3'e^{mt}+C_4'e^{-mt}$ ①

But $\cosh mt=\dfrac{e^{mt}+e^{-mt}}{2}$

$\sinh mt=\dfrac{e^{mt}-e^{-mt}}{2}$

$\cosh mt+\sinh mt=e^{mt}$

$\cosh mt-\sinh mt=e^{-mt}$

∴ ① $\Rightarrow$ $x=C_1\cos mt+C_2\sin mt+C_3'(\cosh mt+\sinh mt)+C_4'(\cosh mt-\sinh mt)$

$x=C_1\cos mt+C_2\sin mt+(C_3'+C_4')\cosh mt+(C_3'-C_4')\sinh mt$

$\boxed{x=C_1\cos mt+C_2\sin mt+C_3\cosh mt+C_4\sinh mt}$

where $C_3=C_3'+C_4'$, $C_4=C_3'-C_4'$

Hence, proved.

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15. Solve: $\dfrac{d^4y}{dx^4}+a^4y=0$

Sol: Given, $\dfrac{d^4y}{dx^4}+a^4y=0$

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$$(D^4 + a^4)\, y = 0$$

The auxiliary equation is:

$$m^4 + a^4 = 0$$

$$(m^2)^2 + (a^2)^2 = 0$$

$$(a+b)^2 = a^2 + b^2 + 2ab$$

$$(m^2 + a^2)^2 - 2a^2m^2 = 0$$

$$\boxed{(a+b)^2 - 2ab = a^2 + b^2}$$

$$(m^2 + a^2)^2 - (\sqrt{2}\,am)^2 = 0$$

$$\boxed{a^2 - b^2 = (a+b)(a-b)}$$

$$(m^2 + a^2 + \sqrt{2}\,am)(m^2 + a^2 - \sqrt{2}\,am) = 0$$

$$m^2 + a^2 + \sqrt{2}\,am = 0$$

$$m^2 + \sqrt{2}\,am + a^2 = 0$$

$$m = \frac{-\sqrt{2}a \pm \sqrt{2a^2 - 4 \cdot 1 \cdot a^2}}{2}$$

$$m = \frac{-\sqrt{2}\,a \pm \sqrt{-2a^2}}{2}$$

$$m = \frac{-\sqrt{2}a \pm \sqrt{2}\,i\,a}{2}$$

$$\boxed{m = \frac{-a}{\sqrt{2}} \pm \frac{ia}{\sqrt{2}}}$$

$$m^2 - \sqrt{2}\,am + a^2 = 0$$

$$m = \frac{\sqrt{2}a \pm \sqrt{2a^2 - 4 \cdot 1 \cdot a^2}}{2}$$

$$m = \frac{\sqrt{2}\,a \pm \sqrt{2}\,i\,a}{2}$$

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$$m = \frac{a}{\sqrt{2}} \pm \frac{ia}{\sqrt{2}}$$

$$y = e^{\frac{-a}{\sqrt{2}}x}\left(C_1 \cos\frac{a}{\sqrt{2}}x + C_2 \sin\frac{a}{\sqrt{2}}x\right) + e^{\frac{a}{\sqrt{2}}x}\left(C_3 \cos\frac{a}{\sqrt{2}}x + C_4 \sin\frac{a}{\sqrt{2}}x\right)$$

16. ☆☆☆ Solve: $(D^4 + 2D^2 + 1)y = 0$

Sol: Given: $(D^4 + 2D^2 + 1)y = 0$

The auxiliary equation is:

$$\Rightarrow m^4 + 2m^2 + 1 = 0$$

$$\Rightarrow (m^2+1)^2 = 0$$

$$\Rightarrow m^2 + 1 = 0$$

$$\Rightarrow m^2 = -1$$

$$\Rightarrow \boxed{m = \pm i, \pm i}$$

$$\boxed{y = (C_1 + C_2 x)\cos x + (C_3 + C_4 x)\sin x}$$

17. Solve: $(D^3 - 5D^2 + 8D - 4)y = 0$

18. Solve: $(D^3 - 1)y = 0$

19. Solve: $(D^3 + D^2 + D)y = 0$

20. Solve: $(D^4 - 8D^2 - 9)y = 0$

21. Solve: $(D^4 - 1)y = 0$

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Solution of non-homogeneous differential equations:

Standard form: $(a_0D^2 + a_1D + a_2)y = f(x)$

Step-1: Solve the corresponding homogeneous differential equation. Here, the solution is known as the complimentary function (CF).

Step-2: Find the particular integral (PI) using

$$P.I = \frac{f(x)}{a_0D^2 + a_1D + a_2}$$

Case-1: Suppose $f(x) = e^{ax+b}$

Put $\boxed{D = a}$

If the denominator becomes zero, then differentiate the denominator and multiply by x.

Case(ii): Suppose $f(x) = \sin(ax+b)$ (or) $\cos(ax+b)$

Put $\boxed{D^2 = -a^2}$

Case(iii): Suppose $f(x)$ is a polynomial,

then we reduce the denominator to any one of the following forms:

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \text{-----}$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \text{-----}$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \text{---}$$

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4. $(1+x)^{-2} = 1-2x+3x^2-4x^3+----$

Step-3: The general solution is: $\boxed{y = CF + PI}$

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1. Solve $y''-6y'+9y = 5e^{-2x}$

Sol: Given: $y''-6y'+9y = 5e^{-2x}$

$$(D^2-6D+9)y = 5e^{-2x}$$

The auxiliary equation is: $m^2-6m+9=0$

$$m^2-3m-3m+9=0$$

$$m(m-3)-3(m-3)=0$$

$$\boxed{m=3,3}$$

Complimentary function (CF) $= (C_1+C_2x)e^{3x}$

Particular integral $= \dfrac{5e^{-2x}}{D^2-6D+9}$

Put $\boxed{D=a=-2}$

$$PI = \frac{5e^{-2x}}{(-2)^2-6(-2)+9}$$

$$PI = \frac{5e^{-2x}}{25}$$

$$\boxed{PI = \frac{e^{-2x}}{5}}$$

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∴ The general solution is : $y = CF + PI$

$$\boxed{y = (c_1 + c_2 x)e^{3x} + \frac{e^{-2x}}{5}}$$

2. Solve: $y'' + 3y' + 2y = 4e^{-x}$

Sol: Given : $y'' + 3y' + 2y = 4e^{-x}$

$$(D^2 + 3D + 2)\, y = 4e^{-x}$$

The auxiliary equation is : $m^2 + 3m + 2 = 0$

$$m^2 + 2m + 1m + 2 = 0$$

$$m(m+2) + 1(m+2) = 0$$

$$(m+1)(m+2) = 0$$

$$\boxed{m = -1, -2}$$

Complimentary function $\boxed{(CF) = c_1 e^{-x} + c_2 e^{-2x}}$

Particular integral $= \dfrac{4e^{-x}}{D^2 + 3D + 2}$

Put $D = a = -1$

$$P.I = \frac{4e^{-x}}{(-1)^2 + 3(-1) + 2}$$

$$P.I = \frac{4e^{-x}}{0}$$

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$$\therefore PI = \frac{4e^{-x}}{2D+3} \cdot x$$

$$= \frac{4e^{-x}}{2(-1)+3} \cdot x$$

$$PI = \frac{4e^{-x}}{1} \cdot x$$

$$\boxed{PI = 4e^{-x} \cdot x}$$

∴ The General solution is :

$$\boxed{y = C_1 e^{-x} + C_2 e^{-2x} + 4e^{-x} \cdot x}$$

3. Solve: $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^x + e^{3x}$

Sol: Given, $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^x + e^{3x}$

$$(D^2 - 6D + 9)y = e^x + e^{3x}$$

The auxiliary equation is : $m^2 - 6m + 9 = 0$

$$m^2 - 3m - 3m + 9 = 0$$

$$m(m-3) - 3(m-3) = 0$$

$$m - 3 = 0, \quad m - 3 = 0$$

$$\boxed{m = 3, 3}$$

∴ Complimentary function (CF) : $y = (C_1 + C_2 x)e^{3x}$

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$$PI = \frac{e^x + e^{3x}}{D^2 - 6D + 9}$$

$$PI = \frac{e^x}{D^2 - 6D + 9} + \frac{e^{3x}}{D^2 - 6D + 9}$$

$\boxed{D = a = 1}$ $\boxed{D = a = 3}$

$$PI = \frac{e^x}{1 - 6 + 9} + \frac{e^{3x}}{9 - 18 + 9}$$

$$PI = \frac{e^x}{4} + \frac{e^{3x}}{0}$$

$$PI = \frac{e^x}{4} + \frac{e^{3x}}{2D - 6} \cdot x$$

$\boxed{D = 3}$

$$PI = \frac{e^x}{4} + \frac{e^{3x}}{0} \cdot x$$

$$\boxed{PI = \frac{e^x}{4} + \frac{e^{3x}}{2} \cdot x^2}$$

∴ The General solution is:

$$y = CF + PI$$

$$\boxed{y = (C_1 + C_2 x)e^{3x} + \frac{e^x}{4} + \frac{e^{3x}}{2} \cdot x^2}$$

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4. Solve: $(D^2-4)y = 3\cosh^2 x$

Sol: Given $(D^2-4)y = 3\cosh^2 x$

The auxiliary equation is: $m^2-4=0$

$$(m)^2 - (2)^2 = 0$$

$$(m+2)(m-2) = 0$$

$$m+2=0, \quad m-2=0$$

$$m=-2, \quad m=2$$

$\therefore$ $\boxed{m=2,-2}$

The Complimentary function $\boxed{(CF): C_1e^{2x} + C_2e^{-2x}}$

$$P.I = \frac{3\cosh^2 x}{D^2-4}$$

$$= \frac{3\left(\frac{e^x+e^{-x}}{2}\right)^2}{D^2-4}$$

$$= \frac{3}{4}\cdot\left\{\frac{e^{2x}+e^{-2x}+2e^x\cdot e^{-x}}{D^2-4}\right\}$$

$$= \frac{3}{4}\left\{\frac{e^{2x}}{D^2-4} + \frac{e^{-2x}}{D^2-4} + \frac{2}{D^2-4}\right\}$$

$D=a=2$ $\quad D=a=-2$ $\quad \boxed{D=0}$

$$= \frac{3}{4}\left\{\frac{e^{2x}}{4-4} + \frac{e^{-2x}}{4-4} + \frac{2}{-4}\right\}$$

$\frac{1}{2}\cdot e^{0x}$

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$$P.I = \frac{3}{4}\left\{\frac{e^{2x}}{2D}\cdot x + \frac{e^{-2x}}{2D}\cdot x + \frac{-2}{4}\right\}$$

Put $\boxed{D=a=2}$ $\boxed{D=a=-2}$

$$= \frac{3}{4}\left\{\frac{e^{2x}}{4}\cdot x + \frac{e^{-2x}\cdot x}{-4} + \left(\frac{-2}{4}\right)\right\}$$

$$P.I = \frac{3}{16}\left\{e^{2x}\cdot x - e^{-2x}\cdot x - 2\right\}$$

The General solution is: $\boxed{y = C_1 e^{2x} + C_2 e^{-2x} + \frac{3}{16}\left\{e^{2x}\cdot x - e^{-2x}\cdot x - 2\right\}}$

5. Solve: $\frac{d^2y}{dx^2} - 4y = \cosh(2x-1) + 3^x$

Sol: Given, $\frac{d^2y}{dx^2} - 4y = \cosh(2x-1) + 3^x$

The auxiliary equation is: $m^2 - 4 = 0$

$$m^2 = 4$$

$$m = \pm\sqrt{4}$$

$$m = \pm 2$$

$\therefore$ $\boxed{m = 2, -2}$

The complimentary function $\boxed{(CF):\ C_1 e^{2x} + C_2 e^{-2x}}$

$$PI = \frac{\cosh(2x-1) + 3^x}{D^2 - 4}$$

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$$PI = \frac{\cosh(2x-1)}{D^2-4} + \frac{3^x}{D^2-4}$$

$3^x = e^{\log 3^x}$

$3^x = e^{x\log 3}$

$$= \frac{\frac{e^{2x-1} + e^{-2x+1}}{2}}{D^2-4} + \frac{e^{x\log 3}}{D^2-4}$$

$$= \frac{1}{2}\left\{\frac{e^{2x-1}}{D^2-4} + \frac{e^{-2x+1}}{D^2-4}\right\} + \frac{e^{x\log 3}}{D^2-4}$$

Put $D=2$ $\quad D=-2$ $\quad D=a=\log 3$

$$= \frac{1}{2}\left\{\frac{e^{2x-1}}{0} + \frac{e^{-2x+1}}{0}\right\} + \frac{e^{x\log 3}}{(\log 3)^2-4}$$

$$= \frac{1}{2}\left\{\frac{e^{2x-1}}{2D}\cdot x + \frac{e^{-2x+1}}{2D}\cdot x\right\} + \frac{e^{x\log 3}}{2\log 3-4}$$

$D=a=2$ $\quad D=a=-2$

$$= \frac{1}{2}\left\{\frac{e^{2x-1}}{4}\cdot x + \frac{e^{-2x+1}}{-4}\cdot x\right\} + \frac{e^{x\log 3}}{(\log 3)^2-4}$$

$$PI = \frac{x}{8}\left\{e^{2x-1} - e^{-2x+1}\right\} + \frac{3^x}{(\log 3)^2-4}$$

The General solution is : $y = CF + PI$

$$y = C_1 e^{2x} + C_2 e^{-2x} + \frac{x}{8}\left\{e^{2x-1} - e^{-2x+1}\right\} + \frac{3^x}{(\log 3)^2-4}$$

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6. Solve: $(D^3 - 6D^2 + 11D - 6)y = e^{-2x} + e^{3x}$

Sol: Given: $(D^3 - 6D^2 + 11D - 6)y = e^{-2x} + e^{3x}$

The auxiliary equation is: $m^3 - 6m^2 + 11m - 6 = 0$

$\boxed{m = 1}$ is a root

1	1	−6	11	−6
	0	1	−5	6
	1	−5	6	0

$\therefore m^2 - 5m + 6 = 0$

$m^2 - 2m - 3m + 6 = 0$

$m(m-2) - 3(m-2) = 0$

$(m-3)(m-2) = 0$

$m = 3, \quad m = 2$

$\therefore \boxed{m = 1, 2, 3}$

The complimentary function $\boxed{(CF): C_1e^x + C_2e^{2x} + C_3e^{3x}}$

$$PI = \frac{e^{-2x} + e^{3x}}{D^3 - 6D^2 + 11D - 6}$$

$$= \frac{e^{-2x}}{D^3 - 6D^2 + 11D - 6} + \frac{e^{3x}}{D^3 - 6D^2 + 11D - 6}$$

Put $\boxed{D = a = -2}$ Put $\boxed{D = a = 3}$

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$$PI = \frac{e^{-2x}}{-8-24-22-6} + \frac{e^{3x}}{27-54+33-6}$$

$$= \frac{e^{-2x}}{-60} + \frac{e^{3x}}{0}$$

$$= \frac{e^{-2x}}{-60} + \frac{e^{3x}}{3D^2-12D+11} \qquad \boxed{D=a=3}$$

$$\boxed{PI = \frac{e^{-2x}}{-60} + \frac{e^{3x}}{2}\cdot x}$$

The General solution is: $\boxed{y = C_1e^{x} + C_2e^{2x} + C_3e^{3x} - \frac{e^{-2x}}{60} + \frac{e^{3x}}{2}x}$

Solve: $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

Sol: Given: $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

$$\{(D+2)(D^2+1-2D)\}y = e^{-2x} + 2\sinh x$$

$$\{D^3 + D - 2D^2 + 2D^2 + 2 - 4D\}y = e^{-2x} + 2\sinh x$$

$$\{D^3 - 3D + 2\}y = e^{-2x} + 2\sinh x$$

The auxiliary equation is: $m^3 - 3m + 2 = 0$

~~$m^3 - 2m - m + 2 = 0$~~ $m^3 - 3m + 2 = 0$

$\boxed{m=1}$ is a root

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1	0	-3	2
0	1	1	-2
1	1	-2	0

$m^2 + m - 2 = 0$

$m^2 + 2m - 1m - 2 = 0$

$m(m+2) - 1(m+2) = 0$

$(m-1)(m+2) = 0$

$m = 1, m = -2$

$\therefore \boxed{m = 1, 1, -2}$

$\therefore \boxed{CF : C_1 e^{-2x} + (C_2 + C_3 x) e^{x}}$

$$PI = \frac{e^{-2x} + 2\sinh x}{D^3 - 3D + 2}$$

$$= \frac{e^{-2x}}{D^3 - 3D + 2} + \frac{2\left\{\frac{e^{x} - e^{-x}}{2}\right\}}{D^3 - 3D + 2}$$

$$= \frac{e^{-2x}}{D^3 - 3D + 2} + \frac{e^{x}}{D^3 - 3D + 2} - \frac{e^{-x}}{D^3 - 3D + 2}$$

Put $D = a = -2$ $\quad D = a = 1$ $\quad D = a = -1$

$$= \frac{e^{-2x}}{0} + \frac{e^{x}}{0} - \frac{e^{-x}}{4}$$

$$= \frac{e^{-2x}}{3D^2 - 3} \cdot x + \frac{e^{x}}{3D^2 - 3} \cdot x - \frac{e^{-x}}{4}$$

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$$PI = \frac{e^{-2x}}{9} \cdot x + \frac{e^{x} \cdot x}{0} - \frac{e^{-x}}{4}$$

$$PI = \frac{e^{-2x}}{9} \cdot x^0 + \frac{e^{x}}{6D} \cdot x^1 - \frac{e^{-x}}{4}$$

$$\boxed{PI = \frac{e^{-2x}}{9} \cdot x + \frac{e^{x}}{6} \cdot x^1 - \frac{e^{-x}}{4}}$$

∴ The General solution is: $\boxed{y = C_1 e^{-2x} + (C_2 + C_3 x)e^{x} + \frac{e^{-2x}}{9} \cdot x + \frac{e^{x}}{6} \cdot x^2 - \frac{e^{-x}}{4}}$

8. Solve: $(y'' + 5y' + 6y) = e^{x} - e^{-2x} + \log 2 + 5^{x}$

9. Solve: $(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 1$

10. Solve: $(D+2)(D+3)(D+4)y = \cosh x$

11. Solve: $(D^3 - 1)y = 21(1 + e^{x})^2$

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12. Solve: $\frac{d^2y}{dx^2} + 5y = \sin 3x$

Sol: Given, $\frac{d^2y}{dx^2} + 5y = \sin 3x$

$$(D^2+5)y = \sin 3x$$

The auxiliary equation is: $m^2+5=0$

$$m^2 = -5$$

$$m = \pm\sqrt{-5}$$

$$\boxed{m = \pm\sqrt{5}\,i}$$

$\therefore$ The complimentary function is $\boxed{(CF): C_1\cos\sqrt{5}x + C_2\sin\sqrt{5}x}$

$$PI = \frac{\sin 3x}{D^2+5}$$

Put $D^2 = -a^2 = -9$

$$PI = \frac{\sin 3x}{-9+5}$$

$$PI = \frac{\sin 3x}{-4}$$

$\therefore$ The General solution is: $y = CF + PI$

$$\boxed{y = C_1\cos\sqrt{5}x + C_2\sin\sqrt{5}x - \frac{\sin 3x}{4}}$$

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13. Solve: $(D^2+4)y = \cos 2x$

Sol: Given, $(D^2+4)y = \cos 2x$

The auxiliary equation is: $m^2+4=0$

$$m^2 = -4$$

$$m = \pm\sqrt{-4}$$

$$m = \pm 2i$$

∴ The complimentary function $\boxed{(CF): C_1\cos 2x + C_2 \sin 2x}$

$$PI = \frac{\cos 2x}{D^2+4}$$

Put $D^2 = a^2 = -4$

$$PI = \frac{\cos 2x}{2D} \cdot x$$

$$= \frac{1}{2}\left(\frac{\cos 2x}{D}\right)\cdot x$$

$$= \frac{1}{2}\left(\frac{\sin 2x}{2}\right)\cdot x$$

$$\boxed{PI = \frac{\sin 2x}{4}\cdot x}$$

(or) $\left(\frac{\cos 2x}{2D}\cdot x\right)\cdot\frac{D}{D}$

$$\frac{-2\sin 2x}{2D^2}\cdot x$$

Put $D^2 = a^2 = -4$

$$\frac{-2\sin 2x}{-8}\cdot x$$

$$\boxed{\frac{\sin 2x}{4}\cdot x}$$

∴ The General solution is:

$$\boxed{y = C_1\cos 2x + C_2\sin 2x + \frac{\sin 2x}{4}\cdot x}$$

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Q: Solve: $(y'' - 5y' - 6y) = 13 \sin 3x$

Sol: Given $(y'' - 5y' - 6y) = 13 \sin 3x$

$$(D^2 - 5D - 6)y = 13 \sin 3x$$

The auxiliary equation is: $m^2 - 5m - 6 = 0$

$$m^2 - 6m + 1m - 6 = 0$$

$$m(m-6) + 1(m-6) = 0$$

$$\boxed{m = 6, \; m = -1}$$

∴ The complimentary function $\boxed{(CF): C_1 e^{-1x} + C_2 e^{6x}}$

$$PI = 13 \cdot \frac{\sin 3x}{D^2 - 5D - 6}$$

Put $D^2 = -a^2 = -9$

$$P.I = \frac{13 \sin 3x}{-9 - 5D - 6}$$

$$P.I = \frac{13 \sin 3x}{-15 - 5D}$$

$$P.I = \frac{13 \sin 3x}{-5(3 + D)} \times \frac{3 - D}{3 - D}$$

$$P.I = 13 \frac{\{3 \sin 3x - 3 \cos 3x\}}{-5(9 - D^2)}$$

$$P.I = \frac{13 \times 3 \{\sin 3x - \cos 3x\}}{-5(10)}$$

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$$P.I = \frac{-13}{30}(\sin 3x - \cos 3x)$$

∴ The General solution is: $y = C_1 e^{-x} + C_2 e^{6x} - \frac{13}{30}(\sin 3x - \cos 3x)$

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15. Solve: $(D^2 - 4D + 3)y = \sin 3x \cdot \cos 2x$

Sol: Given $(D^2 - 4D + 3)y = \sin 3x \cdot \cos 2x$

The auxiliary equation is: $m^2 - 4m + 3 = 0$

$$m^2 - 3m - 1m + 3 = 0$$

$$m(m-3) - 1(m-3) = 0$$

$$m = 1, 3$$

∴ The complimentary function (CF): $C_1 e^{x} + C_2 e^{3x}$

$$P.I = \frac{\sin 3x \cdot \cos 2x}{D^2 - 4D + 3}$$

$$P.I = \frac{\frac{1}{2}\{\sin(5x) + \sin x\}}{D^2 - 4D + 3}$$

$\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$

$$P.I = \frac{1}{2}\left\{\frac{\sin 5x}{D^2 - 4D + 3} + \frac{\sin x}{D^2 - 4D + 3}\right\}$$

$D^2 = -a^2 = -25$ $\quad$ $D^2 = -a^2 = -1$

$$P.I = \frac{1}{2}\left\{\frac{\sin 5x}{-25 - 4D + 3} + \frac{\sin x}{-1 - 4D + 3}\right\}$$

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$$PI = \frac{1}{2}\left\{\frac{\sin 5x}{-4D-22} + \frac{\sin x}{2-4D}\right\}$$

$$PI = \frac{1}{2}\left\{\frac{\sin 5x}{-2(2D+11)} \times \frac{2D-11}{2D-11} + \frac{\sin x}{2(1-2D)} \times \frac{1+2D}{1-2D}\right\}$$

$$PI = \frac{1}{2}\left\{\frac{-11\sin 5x + 2\cdot 5\cos 5x}{-2\{4D^2-121\}}\right\} + \frac{1}{2}\left\{\frac{\sin x + 2\cos x}{2(1-4D^2)}\right\}$$

$D^2 = -a^2 = -25$ $\qquad$ $D^2 = -a^2 = -1$

$$PI = \frac{1}{2}\left\{\frac{-11\sin 5x + 10\cos 5x}{-2\{-221\}}\right\} + \frac{1}{2}\left\{\frac{\sin x + 2\cos x}{2(5)}\right\}$$

$$PI = \frac{1}{4}\left\{\frac{-11\sin 5x + 10\cos 5x}{221}\right\} + \frac{1}{4}\left\{\frac{\sin x + 2\cos x}{5}\right\}$$

∴ The general solution is :

$$y = C_1 e^{x} + C_2 e^{3x} + \frac{1}{4}\left\{\frac{-11\sin 5x + 10\cos 5x}{221}\right\} + \frac{1}{4}\left\{\frac{\sin x + 2\cos x}{5}\right\}$$

16. Solve: $(D^2+4)y = \sin^2 x$

Sol: Given, $(D^2+4)y = \sin^2 x$

The auxiliary equation is : $m^2 + 4 = 0$

$$m^2 = -4$$

$$m = \pm\sqrt{-4}$$

$$\boxed{m = \pm 2i}$$

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∴ The complimentary function (CF): $C_1 \cos 2x + C_2 \sin 2x$

$$PI = \frac{\sin^2 x}{D^2+4}$$

$$PI = \frac{\frac{1-\cos 2x}{2}}{D^2+4}$$

$$PI = \frac{1}{2}\left\{\frac{1-\cos 2x}{D^2+4}\right\}$$

$$PI = \frac{1}{2}\left\{\frac{1}{D^2+4} - \frac{\cos 2x}{D^2+4}\right\}$$

$D=0$; $D^2 = a^2 = -4$

$$PI = \frac{1}{2}\left\{\frac{1}{+4} - \frac{\cos 2x}{-4+4}\right\}$$

$$PI = \frac{1}{2}\left\{\frac{+1}{4} - \frac{\cos 2x}{+0}\right\}$$

$$PI = \frac{1}{2}\left\{\frac{-1}{4} \cdot \frac{\cos 2x}{0}\right\}$$

~~∴ The General solution is~~

$$PI = \frac{1}{2}\left\{\frac{1}{4} - \frac{\sin 2x}{2D \cdot 2} \cdot x\right\}$$

$$PI = \frac{1}{2}\left\{\frac{1}{4} - \frac{\sin 2x}{4} \cdot x\right\}$$

∴ G.S is $y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{2}\left\{\frac{1}{4} - \frac{\sin 2x}{4} \cdot x\right\}$

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Solve: $(D^3-1)y = 3\cos 2x$

Given, $(D^3-1)y = 3\cos 2x$

The auxiliary equation is: $m^3-1=0$

$$m^3 = 1$$

$$\boxed{m = 1}$$

	1	0	0	-1
1		1	1	1
	1	1	1	0

$$m^2+m+1=0$$

$$m = \frac{-1 \pm \sqrt{1-4\cdot 1\cdot 1}}{2}$$

$$m = \frac{-1 \pm \sqrt{-3}}{2}$$

$$m = \frac{-1 \pm \sqrt{3}i}{2}$$

$\therefore$ The complimentary function $\boxed{(CF): C_1 e^{x} + e^{-\frac{x}{2}}\left(C_2 \cos\frac{\sqrt{3}}{2}x + C_3 \sin\frac{\sqrt{3}}{2}x\right)}$

$$PI = \frac{3\cos 2x}{D^3-1}$$

$$PI = \frac{3\cos 2x}{D^2\cdot D - 1} \qquad D^2 = a^2 = -4$$

$$PI = \frac{3\cos 2x}{-4D-1}$$

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$$PI = \frac{3\cos 2x}{-1(1+4D)} \times \frac{1-4D}{1-4D}$$

$$PI = \frac{3\cos 2x - 12\cdot(-2\sin 2x)}{-1(1-16D^2)}$$

$$PI = \frac{3\cos 2x + 24\sin 2x}{-1+16D^2}$$

$$D^2 = a^2 = -4$$

$$PI = \frac{3\cos 2x + 24\sin 2x}{-1+16(-4)}$$

$$PI = \frac{3\cos 2x + 24\sin 2x}{-65}$$

∴ The General solution is: $\boxed{y = C_1 e^{x} + e^{\frac{-x}{2}}\left(C_2\cos\frac{\sqrt{3}}{2}x + C_3\sin\frac{\sqrt{3}}{2}x\right) + \frac{3\cos 2x+24\sin 2x}{-65}}$

18. Solve: $(D^2+a^2)y = \cos ax$

Sol: Given, $(D^2+a^2)y = \cos ax$

The auxiliary equation is: $m^2+a^2=0$

$$m^2 = -a^2$$

$$m = \pm\sqrt{-a^2}$$

$$\boxed{m = \pm ia}$$

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∴ Complimentary function (CF): $C_1 \cos ax + C_2 \sin ax$

$$P.I = \frac{\cos ax}{D^2 + a^2}$$

Put $D^2 = -a^2$

$$P.I = \frac{\cos ax}{-a^2 + a^2}$$

$$P.I = \frac{\cos ax}{0}$$

$$P.I = \frac{\cos ax}{2D} \cdot x$$

$$= \frac{1}{2}\left(\frac{\sin ax}{a}\right) \cdot x$$

$$P.I = \frac{\sin ax}{2a} \cdot x$$

∴ The General solution is: $\boxed{y = C_1 \cos ax + C_2 \sin ax + \frac{\sin ax \cdot x}{2a}}$

19. Solve: $(D^2 + a^2)y = \sin ax$

20. Solve: $(D^2 + 8D + 25)y = 48 \cos x - 16 \sin x$

21. Solve: $\left(\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx}\right) = e^{-x} + \sin 2x$

22. $(D^4 - 2D^2 + 1)y = \cos x$

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23. Solve: $(y''-7y'+6y) = 1-x+x^2$

Sol: Given, $(y''-7y'+6y) = 1-x+x^2$

The auxiliary equation is $m^2-7m+6=0$

$$m^2-6m-1m+6=0$$

$$m(m-6)-1(m-6)=0$$

$$\boxed{m=6,1}$$

∴ The complimentary function $\boxed{(CF): C_1e^{6x}+C_2e^{x}}$

$$P.I = \frac{1-x+x^2}{D^2-7D+6}$$

$$P.I = \frac{1-x+x^2}{6\left\{1+\frac{D^2-7D}{6}\right\}}$$

$$P.I = \frac{1}{6}\left\{1+\frac{D^2-7D}{6}\right\}^{-1}(1-x+x^2) \qquad \left[\because (1+x)^{-1} = 1-x+x^2-x^3+\cdots\right]$$

$$P.I = \frac{1}{6}\left\{1-\left(\frac{D^2-7D}{6}\right)+\left(\frac{D^2-7D}{6}\right)^2\right\}(1-x+x^2)$$

$$P.I = \frac{1}{6}\left\{1-\frac{D^2}{6}+\frac{7D}{6}+\frac{49D^2}{36}\right\}(1-x+x^2)$$

$$P.I = \frac{1}{6\times 36}\left\{36-6D^2+42D+49D^2\right\}(1-x+x^2)$$

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$$P.I = \frac{1}{216}\left\{36+42D+43D^2\right\}\left(1-x+x^2\right)$$

$$P.I = \frac{1}{216}\left\{36-36x+36x^2+42(-1+2x)+43(2)\right\}$$

$$P.I = \frac{1}{216}\left\{36-36x+36x^2-42+84x+86\right\}$$

$$P.I = \frac{1}{216}\left\{36x^2+48x+80\right\}.$$

∴ The General solution is: $y = C_1e^{6x}+C_2e^{x}+\frac{1}{216}\left\{36x^2+48x+80\right\}$

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24. Solve: $(D^2+4D-3)y = 2x^2+3x-1$

Sol: Given, $(D^2+4D-3)y = 2x^2+3x-1$

The auxiliary equation is :- $m^2+4m-3=0$

$$m = \frac{-4\pm\sqrt{16-4\cdot1\cdot(-3)}}{2}$$

$$m = \frac{-4\pm\sqrt{16+12}}{2}$$

$$m = \frac{-4\pm\sqrt{28}}{2}$$

$$m = \frac{-4\pm2\sqrt{7}}{2}$$

$$\boxed{m = -2\pm\sqrt{7}}$$

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∴ The complimentary function (CF): $C_1 e^{(-2+\sqrt{7})x} + C_2 e^{(-2-\sqrt{7})x}$

$$P.I = \frac{2x^2+3x-1}{D^2+4D-3}$$

$$P.I = \frac{2x^2+3x-1}{-3\left(1-\left(\frac{D^2+4D}{3}\right)\right)}$$

$$P.I = \frac{-1}{3}\left\{1-\left(\frac{D^2+4D}{3}\right)\right\}^{-1}\cdot\left(2x^2+3x-1\right)$$

$$P.I = \frac{-1}{3}\left\{1+\left(\frac{D^2+4D}{3}\right)+\left(\frac{D^2+4D}{3}\right)^2\right\}\left(2x^2+3x-1\right)$$

$$P.I = \frac{-1}{3}\left\{1+\frac{D^2+4D}{3}+\frac{16D^2}{9}\right\}\left(2x^2+3x-1\right)$$

$$P.I = \frac{-1}{3}\left\{1+\frac{D^2}{3}+\frac{4D}{3}+\frac{16D^2}{9}\right\}\left(2x^2+3x-1\right)$$

$$= \frac{-1}{27}\left\{9+3D^2+12D+16D^2\right\}\left(2x^2+3x-1\right)$$

$19D^2$

$$= \frac{-1}{27}\left\{18x^2+27x-9+12\{4x+3\}+19\{4\}\right\}$$

$$= \frac{-1}{27}\left\{18x^2+27x+48x-9+36+76\right\}$$

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$$P.I = \frac{-1}{27}\left\{18x^2 + 75x + 103\right\}$$

∴ The General solution is:

$$y = C_1 e^{(-2+\sqrt{7})x} + C_2 e^{(-2-\sqrt{7})x} + \left(\frac{-1}{27}\right)\left\{18x^2 + 75x + 103\right\}$$

25. Solve: $(y'' + 4y' + 3y) = x^3$

Sol: Given, $(y'' + 4y' + 3y) = x^3$

The auxiliary equation is: $m^2 + 4m + 3 = 0$

$$m^2 + 3m + 1m + 3 = 0$$

$$m(m+3) + 1(m+3) = 0$$

$$(m+1)(m+3) = 0$$

$$\boxed{m = -1, -3}$$

∴ The complimentary function (CF): $C_1 e^{-x} + C_2 e^{-3x}$

$$P.I = \frac{x^3}{(D^2 + 4D + 3)}$$

$$= \frac{x^3}{3\left(1 + \frac{D^2 + 4D}{3}\right)}$$

$$= \frac{1}{3}\left\{1 + \frac{D^2 + 4D}{3}\right\}^{-1} \cdot (x^3)$$

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$$PI = \frac{1}{3}\left\{1 - \left(\frac{D^2+4D}{3}\right) + \left(\frac{D^2+4D}{3}\right)^2 - \left(\frac{D^2+4D}{3}\right)^3\right\}(x^3)$$

$$PI = \frac{1}{3}\left\{1 - \frac{D^2}{3} - \frac{4D}{3} + \frac{16D^2}{9} + \frac{8D^3}{9} - \frac{64D^3}{27}\right\}x^3$$

$$PI = \frac{1}{3\times 27}\left\{27 - 9D^2 - 36D + 48D^2 + 24D^3 - 64D^3\right\}x^3$$

$$PI = \frac{1}{81}\left\{27 - 36D + 39D^2 - 40D^3\right\}x^3$$

$$PI = \frac{1}{81}\left\{27x^3 - 36(3x^2) + 39(6x) - 40(6)\right\}$$

$$PI = \frac{1}{81}\left(27x^3 - 108x^2 + 234x - 240\right)$$

. The General solution is:

$$y = C_1e^{-x} + C_2e^{-3x} + \frac{1}{81}\left(27x^3 - 108x^2 + 234x - 240\right)$$

*****

26. Solve: $(y'' + y') = x^2 + 2x + 1$

Sol: Given $(y'' + y') = x^2 + 2x + 1$

The auxiliary equation is $m^2 + m = 0$

$$m(m+1) = 0$$

$$m = 0, -1$$

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∴ The complimentary function $\boxed{(CF): C_1 + C_2 e^{-x}}$

$$P.I = \frac{x^2+2x+1}{D^2+D}$$

$$P.I = \frac{x^2+2x+1}{D(D+1)}$$

$$P.I = \frac{x^2+2x+1}{D(1+D)}$$

$$P.I = \frac{1}{D}\left\{(1+D)^{-1}\right\}(x^2+2x+1)$$

$$P.I = \frac{1}{D}\left\{1-D+D^2\right\}(x^2+2x+1)$$

$$P.I = \frac{1}{D}\left\{x^2+2x+1-(2x+2)+2\right\}$$

$$P.I = \frac{1}{D}\left\{x^2+2x+1-2x-2+2\right\}$$

$$P.I = \frac{1}{D}\left\{x^2+1\right\}$$

$$\boxed{P.I = \frac{x^3}{3}+x}$$

∴ The General solution is: $\boxed{y = C_1 + C_2 e^{-x} + \frac{x^3}{3} + x}$

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27. Solve: $(y''-6y'+9y) = 54x+18$

Sol: Given, $(y''-6y'+9y) = 54x+18$

The auxiliary equation is: $m^2-6m+9=0$

$$m^2-3m-3m+9=0$$

$$m(m-3)-3(m-3)=0$$

$$\boxed{m=3,3}$$

$\therefore$ The complimentary function $\boxed{(CF): (C_1+C_2x)e^{3x}}$

$$P.I = \frac{54x+18}{D^2-6D+9}$$

$$= \frac{54x+18}{9\left(1+\frac{D^2-6D}{9}\right)}$$

$$= \frac{1}{9}\left(1+\frac{D^2-6D}{9}\right)^{-1}\cdot(54x+18)$$

$$= \frac{1}{9}\left\{1-\left(\frac{D^2-6D}{9}\right)\right\}(54x+18)$$

$$= \frac{1}{9}\left\{1-\frac{D^2}{9}+\frac{6D}{9}\right\}(54x+18)$$

$$= \frac{1}{9\times 9}\left\{9-D^2+6D\right\}(54x+18)$$

$$= \frac{1}{81}\left\{486x+162+6(54)\right\}$$

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$$P.I = \frac{1}{81}\{486 + 486x\}$$

$$= \frac{486}{81}\{1+x\}$$

$$= 6\{1+x\}$$

$$\boxed{PI = 6+6x}$$

∴ The General solution is: $\boxed{y = (C_1 + C_2 x)e^{3x} + 6x + 6}$

28. Solve: $(D^3 - 3D - 2)y = x^2$

Sol: Given, $(D^3 - 3D - 2)y = x^2$

The auxiliary equation is: $m^3 - 3m - 2 = 0$

$\boxed{m = -1}$ is a root

-1	1	0	-3	-2
	0	-1	1	$+2$
	1	-1	-2	0

$$m^2 - m - 2 = 0$$

$$m^2 - 2m + m - 2 = 0$$

$$m(m-2) + 1(m-2) = 0$$

$$(m-2)(m+1) = 0$$

$$\boxed{m = 2, -1}$$

∴ $m = -1, -1, 2$

∴ The complimentary function $\boxed{(CF) = (C_1 + C_2 x)e^{-x} + C_3 e^{2x}}$

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$$P.I = \frac{x^2}{D^3 - 3D - 2}$$

$$= \frac{x^2}{-2\left(1 - \left(\frac{D^3 - 3D}{2}\right)\right)}$$

$$= \frac{-1}{2}\left(1 - \left(\frac{D^3 - 3D}{2}\right)\right)^{-1} \cdot x^2$$

$$= \frac{-1}{2}\left\{1 + \left(\frac{D^3 - 3D}{2}\right) + \left(\frac{D^3 - 3D}{2}\right)^2\right\} \cdot x^2$$

$$= \frac{-1}{2 \times 8}\left\{2 + \cancel{D^3} - 3D + \frac{9D^2}{4}\right\} \cdot x^2$$

$$= \frac{-1}{2}\left\{1 - \frac{3D}{2} + \frac{9D^2}{4}\right\} \cdot x^2$$

$$= \frac{-1}{2 \times 4}\left\{4 - 6D + 9D^2\right\} \cdot x^2$$

$$= \frac{-1}{8}\left\{4x^2 - 6\{2x\} + 9\{2\}\right\}$$

$$= \frac{-1}{8}\left\{4x^2 - 12x + 18\right\}$$

∴ The General solution is: $y = (C_1 + C_2 x)e^{-x} + C_3 e^{2x} - \frac{1}{6}\left\{4x^2 - 12x + 18\right\}$

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Solve: $(D^2-4D+4)y = 8(e^{2x} + \sin 2x + x^2)$

Sol: Given, $(D^2-4D+4)y = 8(e^{2x} + \sin 2x + x^2)$

The auxiliary equation is: $m^2-4m+4=0$

$$m^2-2m-2m+4=0$$

$$m(m-2)-2(m-2)=0$$

$$(m-2)(m-2)=0$$

$$\boxed{m=2,2}$$

∴ The complimentary function (CF): $(C_1+C_2x)e^{2x}$

$$PI = 8\left\{\frac{e^{2x}+\sin 2x+x^2}{D^2-4D+4}\right\}$$

$$= 8\left\{\frac{e^{2x}}{D^2-4D+4} + \frac{\sin 2x}{D^2-4D+4} + \frac{x^2}{D^2-4D+4}\right\}$$

$$P.I = PI_1 + PI_2 + PI_3$$

$$PI_1 = \frac{8e^{2x}}{D^2-4D+4}$$

Put $D=0=2$

$$= \frac{8e^{2x}}{4-4(2)+4}$$

$$= \frac{8e^{2x}}{0}$$

$$P.I_1 = \frac{8 \cdot e^{2x}}{2D-4} \cdot x$$

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$$P.I_1 = 8 \cdot \frac{e^{2x}}{0} \cdot x$$

$$P.I_1 = 8 \cdot \frac{e^{2x}}{2} \cdot x^2$$

$$P.I_1 = 4 \cdot e^{2x} \cdot x^2$$

where $P.I_2 = \dfrac{8 \sin 2x}{D^2 - 4D + 4}$

$$P.I_2 = \frac{8\{2 \sin x \cos x\}}{D^2 - 4D + 4}$$

$$P.I_2 = \frac{16 \sin x \cos x}{D^2 - 4D + 4}$$

$$P.I_2 = \frac{8 \sin 2x}{D^2 - 4D + 4}$$

$$D^2 = -a^2 = -4$$

$$= 8 \frac{\sin 2x}{-4 - 4D + 4}$$

$$= 8 \frac{\sin 2x}{-4D}$$

$$= -2 \left(\frac{-\cos 2x}{2}\right)$$

$$P.I_2 = \cos 2x$$

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$$P.I_3 = \frac{8x^2}{D^2-4D+4}$$

$$= 8 \cdot \frac{x^2}{4\left(1+\frac{D^2-4D}{4}\right)}$$

$$= \frac{8}{4} \cdot \left(1+\frac{D^2-4D}{4}\right)^{-1} \cdot x^2$$

$$= 2\left\{1-\left(\frac{D^2-4D}{4}\right)+\left(\frac{D^2-4D}{4}\right)^2\right\} \cdot x^2$$

$$= 2\left\{1-\frac{D^2}{4}+\frac{4D}{4}+\frac{16D^2}{16}\right\} \cdot x^2$$

$$= \frac{2}{16}\left\{16-4D^2+16D+16D^2\right\} \cdot x^2$$

$$= \frac{2}{16}\left\{16+16D+12D^2\right\} \cdot x^2$$

$$= \frac{2}{16}\left\{16x^2+16(2x)+12(2)\right\}$$

$$= \frac{2}{16}\left\{16x^2+32x+24\right\}$$

$$= 2x^2+4x+3$$

(or)

$$P.I_3 = \frac{8x^2}{(2-D)^2}$$

$$= \frac{8x^2}{4\left(1-\frac{D}{2}\right)^2}$$

$$= \left(1-\frac{D}{2}\right)^{-2} \cdot 2x^2$$

$$= \left\{1+2\left(\frac{D}{2}\right)+3\left(\frac{D^2}{4}\right)\right\} \cdot x^2$$

$$\therefore P.I = P.I_1 + P.I_2 + P.I_3 = 4e^{2x} \cdot x^2 + \cos 2x + 2x^2+4x+3$$

∴ The General solution is:

$$\boxed{y = (C_1+C_2x)e^{2x} + 4x^2e^{2x} + \cos 2x + 2x^2+4x+3}$$

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30. Solve: $(y''-4y) = x^2 + \cos 2x + 2^{-x}$

Sol: Given, $(y''-4y) = x^2 + \cos 2x + 2^{-x}$

The auxiliary equation is: $m^2 - 4 = 0$

$$m^2 = 4$$

$$m = \pm\sqrt{4}$$

$$\boxed{m = \pm 2}$$

The complimentary function $\boxed{(CF): C_1 e^{2x} + C_2 e^{-2x}}$

$$P.I = \frac{x^2 + \cos 2x + 2^{-x}}{D^2 - 4}$$

$$P.I = \frac{x^2}{D^2-4} + \frac{\cos 2x}{D^2-4} + \frac{2^{-x}}{D^2-4}$$

$$P.I = P.I_1 + P.I_2 + P.I_3$$

where $P.I_1 = \frac{x^2}{D^2-4}$

$$P.I_1 = \frac{x^2}{-4\left(1 - \frac{D^2}{4}\right)}$$

$$P.I_1 = \frac{-1}{4} \cdot \left(1 - \frac{D^2}{4}\right)^{-1} \cdot x^2$$

$$P.I_1 = \frac{-1}{4} \cdot \left(1 + \left(\frac{D^2}{4}\right) + \left(\frac{D^2}{4}\right)^2\right) \cdot x^2$$

$$P.I_1 = \frac{-1}{4}\left(1 + \frac{D^2}{4}\right) \cdot x^2$$

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$$P.I_1 = \frac{-1}{16}\left(4 + D^2\right)\cdot x^2$$

$$P.I_1 = \frac{-1}{16}\left(4x^2 + 2\right)$$

$$P.I_1 = -\frac{1}{4}x^2 - \frac{1}{8}$$

$$P.I_2 = \frac{\cos 2x}{D^2 - 4}$$

Put $D^2 = -a^2 = -4$

$$= \frac{\cos 2x}{-4-4}$$

$$P.I_2 = \frac{-\cos 2x}{8}$$

$$P.I_3 = \frac{2^{-x}}{D^2-4}$$

$$2^{-x} = e^{\log_2 -x}$$

$$2^{-x} = e^{-x\log 2}$$

$$P.I_3 = \frac{e^{-x\log 2}}{D^2-4}$$

e^{ax} where $a = -\log 2$

$D = a = -\log 2$

$$P.I_3 = \frac{e^{-x\log 2}}{(-\log 2)^2 - 4}$$

$$P.I_3 = \frac{2^{-x}}{(-\log 2)^2 - 4}$$

$\therefore P.I = P.I_1 + P.I_2 + P.I_3$

$$P.I = -\frac{x^2}{4} - \frac{1}{8} - \frac{\cos 2x}{8} + \frac{2^{-x}}{(-\log 2)^2 - 4}$$

$\therefore$ General solution is: $y = C_1 e^{2x} + C_2 e^{-2x} - \frac{x^2}{4} - \frac{1}{8} - \frac{\cos 2x}{8} + \frac{2^{-x}}{(-\log 2)^2 - 4}$

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31. Solve: $(y''+y) = e^{-x} + \cos x + x^3$

32. Solve: $(y''-6y'+25y) = e^{2x} + \sin x + x$

33. Solve: $(D^3-3D^2+4D-2)y = e^x + \cos x$

34. Solve: $(D^4-1)y = \cosh(x-1) + 2\sin x$

Cauchy's differential equation:

A differential equation of the form:

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n y = f(x)$$

is called Cauchy's differential equation.

Put $z = \log x \Rightarrow x = e^z$

$\therefore x \frac{dy}{dx} = Dy$ where $D = \frac{d}{dz}$

$$x^2 \frac{d^2 y}{dx^2} = D(D-1)y$$

$$x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)y \quad \dots$$

Legendre's differential equation:

A differential equation of the form:

$$(ax+b)^n \frac{d^n y}{dx^n} + a_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n y = f(x)$$

is called Legendre's differential equation.

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Put $z = \log(ax+b)$

$\therefore (ax+b)\frac{dy}{dx} = aDy$ where $D = \frac{d}{dz}$

$(ax+b)^2 \frac{d^2y}{dx^2} = a^2 D(D-1)y$

$(ax+b)^3 \frac{d^3y}{dx^3} = a^3 D(D-1)(D-2)y$.

1. Solve: $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$

***
RHS must be in terms of z

Sol: Given, $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4 \rightarrow (1)$

Put $z = \log x \Rightarrow x = e^z$

$\frac{dz}{dx} = \frac{1}{x}$

$\frac{dz}{dy}\cdot\frac{dy}{dx} = \frac{1}{x}$

$x\frac{dy}{dx} = \frac{dy}{dz}$

$\therefore x\frac{dy}{dx} = Dy, \quad D = \frac{d}{dz}$

$x^2\frac{d^2y}{dx^2} = D(D-1)y$

$(1) \Rightarrow (D(D-1) - 2D - 4)y = (e^z)^4$

$(D^2 - 3D - 4)y = (e^z)^4$.

The auxiliary equation is: $m^2 - 3m - 4 = 0$

$m^2 - 4m + 1m - 4 = 0$

$m(m-4) + 1(m-4) = 0$

$(m+1)(m-4) = 0$

$\boxed{m = 4, -1}$

The complementary function $\boxed{(CF): C_1e^{4z} + C_2e^{-z}}$

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$$PI = \frac{e^{4z}}{D^2-3D-4}$$

Put $D = a = 4$

$$PI = \frac{e^{4z}}{16-12-4}$$

$$PI = \frac{e^{4z}}{0}$$

$$PI = \frac{e^{4z}}{2D-3} \cdot z$$

$D = a = 4$

$$\boxed{PI = \frac{e^{4z}}{5} \cdot z}$$

$\therefore$ The General solution is : $y = c_1 e^{4z} + c_2 e^{-z} + \frac{e^{4z}}{5} \cdot z$

$$\therefore y = c_1 e^{4\log x} + c_2 e^{-(\log x)} + \frac{e^{4(\log x)}}{5} \cdot \log x$$

$$y = c_1 e^{\log x^4} + c_2 e^{\log x^{-1}} + \frac{e^{\log x^4}}{5} \cdot \log x$$

$$\boxed{y = c_1 x^4 + c_2 x^{-1} + \frac{x^4 \log x}{5}}$$

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Solve: $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = 3\sin(\log x)$

Sol: Given, $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = 3\sin(\log x) \longrightarrow (1)$

Put $z = \log x \Rightarrow x = e^z$

$$x\frac{dy}{dx} = Dy,$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$(1) \Rightarrow D(D-1)y - 3Dy + 5y = 3\sin(\log x)$

$$\left[D(D-1) - 3D + 5\right]y = 3\sin(\log x)$$

$$\left[D^2 - D - 3D + 5\right]y = 3\sin z$$

$$\left[D^2 - 4D + 5\right]y = 3\sin z$$

The auxiliary equation is: $m^2 - 4m + 5 = 0$

$$m = \frac{4 \pm \sqrt{16 - 4\cdot 1\cdot 5}}{2}$$

$$m = \frac{4 \pm \sqrt{16 - 20}}{2}$$

$$m = \frac{4 \pm \sqrt{-4}}{2} \Rightarrow m = \frac{4 \pm 2i}{2}$$

$$\therefore \boxed{m = 2 \pm i}$$

$\therefore$ Complimentary function $\boxed{(CF): e^{2x}(c_1\cos z + c_2\sin z)}$

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$$P.I = \frac{3\sin z}{D^2-4D+5}$$

Put $D^2 = -a^2 = -1$

$$P.I = 3\left\{\frac{\sin z}{-1-4D+5}\right\}$$

$$= 3\left(\frac{\sin z}{4-4D}\right)$$

$$= 3\left\{\frac{\sin z}{4(1-D)} \times \frac{1+D}{1+D}\right\}$$

$$= \frac{3}{4}\left\{\frac{\sin z + \cos z}{1-D^2}\right\}$$

$$= \frac{3}{4}\left\{\frac{\sin z}{1-D^2} + \frac{\cos z}{1-D^2}\right\}$$

$D^2 = -a^2 = -1$ $\quad$ $D^2 = -a^2 = -1$

$$= \frac{3}{4}\left\{\frac{\sin z}{1-(-1)} + \frac{\cos z}{1-(-1)}\right\}$$

$$= \frac{3}{4}\left\{\frac{\sin z + \cos z}{2}\right\}$$

$$= \frac{3}{8}\left\{\sin z + \cos z\right\}$$

∴ The General solution is:

$$y = e^{2z}\left(C_1\cos z + C_2 \sin z\right) + \frac{3}{8}\left(\sin z + \cos z\right)$$

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$$y = e^{\frac{1}{2}\log x}\left(C_1\cos(\log x) + C_2\sin(\log x)\right) + \frac{3}{8}\left(\sin(\log x) + \cos(\log x)\right)$$

$$y = e^{\log x^{\frac{1}{2}}}\left(C_1\cos(\log x) + C_2\sin(\log x)\right) + \frac{3}{8}\left(\sin(\log x) + \cos(\log x)\right)$$

$$y = x^{\frac{1}{2}}\left[C_1\cos(\log x) + C_2\sin(\log x)\right] + \frac{3}{8}\left[\sin(\log x) + \cos(\log x)\right]$$

3. Solve: $x^2y'' - xy' + y = \log x$

Sol: Given, $x^2y'' - xy' + y = \log x \rightarrow (1)$

Put $z = \log x \Rightarrow x = e^z$

$$x\frac{dy}{dx} = Dy$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$(1) \Rightarrow \left[D(D-1) - D + 1\right] y = \log x$

$$\left[D^2 - D - D + 1\right]y = \log x$$

$$\left[D^2 - 2D + 1\right]y = \log x$$

$$\left[D^2 - 2D + 1\right]y = z$$

The auxiliary equation is: $m^2 - 2m + 1 = 0$

$$m^2 - 1m - 1m + 1 = 0$$

$$m(m-1) - 1(m-1) = 0$$

$$\boxed{m = 1, 1}$$

∴ The complimentary function $\boxed{(CF): (C_1 + C_2 z)e^z}$

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$$P.I = \frac{Z}{D^2-2D+1}$$

$$= \frac{Z}{1\left[1+\frac{D^2-2D}{1}\right]}$$

$$= \left[1+\frac{D^2-2D}{1}\right]^{-1} \cdot Z$$

$$= \left[1-(D^2-2D)\right] \cdot Z$$

$$= \left[1-D^2+2D\right] \cdot Z$$

$$= \left[1+2D\right] \cdot Z$$

$$P.I = \left[Z+2\right]$$

∴ The general solution is:

$$y = (C_1+C_2 Z)e^Z + (Z+2)$$

$$= \left(C_1+C_2(\log x)\right)e^{\log x} + (\log x+2)$$

$$\boxed{y = (C_1+C_2 \log x)\,x + (\log x+2)}$$

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Solve: $x\frac{d^2y}{dx^2} - \frac{2y}{x} = x + \frac{1}{x^2}$

Sol: Given, $x\frac{d^2y}{dx^2} - \frac{2y}{x} = x + \frac{1}{x^2}$

Multiply throughout by x

$$x^2\frac{d^2y}{dx^2} - 2y = x^2 + \frac{1}{x} \longrightarrow (1)$$

Put $z = \log x \Rightarrow x = e^z$

$$x\frac{dy}{dx} = Dy$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$\{D(D-1) - 2\}y = (e^z)^2 + \frac{1}{e^z}$$

$$\{D^2 - D - 2\}y = e^{2z} + e^{-z}$$

The auxiliary equation is: $m^2 - m - 2 = 0$

$$m^2 - 2m + 1m - 2 = 0$$

$$m(m-2) + 1(m-2) = 0$$

$$(m+1)(m-2) = 0$$

$$\boxed{m = 2, -1}$$

∴ The complimentary function $\boxed{(CF): C_1e^{2x} + C_2e^{-x}}$

$$P.I = \frac{e^{2z}}{D^2 - D - 2} + \frac{e^{-z}}{D^2 - D - 2}$$

$$D = a = 2 \qquad D = a = -1$$

$$P.I = \frac{e^{2z}}{0} + \frac{e^{-z}}{0}$$

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$$\therefore P.I = \frac{e^{2z}}{2D-1} \cdot z + \frac{e^{-z}}{2D-1} \cdot z$$

$D=a=2$ $\qquad$ $D=a=-1$

$$= \frac{e^{2z}}{3} \cdot z + \frac{e^{-z}}{-3} \cdot z$$

$$= \frac{e^{2z}}{3} \cdot z - \frac{e^{-z}}{3} \cdot z$$

$\therefore$ The General solution is:

$$y = C_1 e^{2z} + C_2 e^{-z} + \frac{e^{2z}}{3} \cdot z - \frac{e^{-z}}{3} \cdot z$$

$$y = C_1 e^{2z} + C_2 e^{-z} + \frac{e^{2(\log x)}}{3} \cdot \log x - \frac{e^{-(\log x)}}{3} \cdot \log x$$

$$y = C_1 e^{2(\log x)} + C_2 e^{-(\log x)} + \frac{e^{\log x^2}}{3} \cdot \log x - \frac{e^{\log x^{-1}}}{3} \cdot \log x$$

$$\boxed{y = C_1 x^2 + C_2 x^{-1} + \frac{x^2 \log x}{3} - \frac{1}{3x} \log x}$$

5. Solve: $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$

Sol: Given $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right) \rightarrow (1)$

$$\left(x^3 D^3 + 2x^2 D^2 + 2\right) y = 10\left(x + \frac{1}{x}\right)$$

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Put $z = \log x \Rightarrow x = e^z$

$$x\frac{dy}{dx} = Dy$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

$$\{D(D-1)(D-2) + 2D(D-1) + 2\}y = 10\left(x+\frac{1}{x}\right)$$

$$\{D(D^2-3D+2) + 2D^2 - 2D + 2\}y = 10\left(x+\frac{1}{x}\right)$$

$$\{D^3 - 3D^2 + 2D + 2D^2 - 2D + 2\}y = 10\left(x+\frac{1}{x}\right)$$

$$\{D^3 - D^2 + 2\}y = 10\left(x+\frac{1}{x}\right)$$

The auxiliary equation is : $m^3 - m^2 + 2 = 0$

$\boxed{m=-1}$ is a root

-1	1	-1	0	2
	0	-1	2	-2
	1	-2	2	$\boxed{0}$

$$m^2 - 2m + 2 = 0$$

$$m = \frac{2 \pm \sqrt{4 - 4\cdot 1\cdot 2}}{2}$$

$$m = \frac{2 \pm \sqrt{-4}}{2}$$

$$m = \frac{2 \pm 2i}{2} \Rightarrow \boxed{m = 1 \pm i}$$

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∴ The complimentary function (CF) : $C_1 e^{-x} + e^{x}(C_2 \cos z + C_3 \sin z)$

$$P.I = \frac{10\left(x + \frac{1}{x}\right)}{D^3 - D^2 + 2}$$

$$= \frac{10x}{D^3 - D^2 + 2} + \frac{10 \cdot x^{-1}}{D^3 - D^2 + 2}$$

$$= \frac{10 \cdot e^{z}}{D^3 - D^2 + 2} + \frac{10 \cdot e^{-z}}{D^3 - D^2 + 2}$$

$$= 10\left\{\frac{e^{z}}{D^3 - D^2 + 2} + \frac{e^{-z}}{D^3 - D^2 + 2}\right\}$$

$D = a = 1$ $\qquad$ $D = a = -1$

$$= 10\left\{\frac{e^{z}}{2} + \frac{e^{-z}}{0}\right\}$$

$$= 10\left\{\frac{e^{z}}{2} + \frac{e^{-z}}{3D^2 - 2D} \cdot z\right\}$$

$D = a = -1$

$$= 10\left\{\frac{e^{z}}{2} + \frac{e^{-z}}{5} \cdot z\right\}$$

$$P.I = 5e^{z} + 2e^{-z} \cdot z$$

∴ General solution is : $y = C_1 e^{-\log x} + e^{\log x}(C_2 \cos \log x + C_3 \sin(\log x)) + 5e^{\log x} + 2e^{-\log x} \cdot \log x$

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$$y = C_1 \cdot x^{-1} + x\left(C_2 \cos \log x + C_3 \sin \log x\right) + 5x + 2x^{-1} \log x$$

6. Solve: $(2x+1)^2 \cdot y'' - 6(2x+1) \cdot y' + 16y = 8(2x+1)^2$

Sol: Given, $(2x+1)^2 \cdot y'' - 6(2x+1) \cdot y' + 16y = 8(2x+1)^2 \longrightarrow (1)$

Put $z = \log(ax+b)$

$$z = \log(2x+1) \Rightarrow 2x+1 = e^z$$

$$(2x+1)\frac{dy}{dx} = 2Dy, \quad D = \frac{d}{dz}$$

$$(2x+1)^2 \frac{d^2y}{dx^2} = 4D(D-1)y$$

$\therefore (1) \Rightarrow \{4D(D-1) - 12D + 16\} y = 8(2x+1)^2$

$$\{4(D^2-D) - 12D + 16\} y = 8 \cdot (e^z)^2$$

$$\{4D^2 - 4D - 12D + 16\} y = 8e^{2z}$$

$$\{4D^2 - 16D + 16\} y = 8e^{2z}$$

$$4\{D^2 - 4D + 4\} y = 8e^{2z}$$

$$\{D^2 - 4D + 4\} y = 2e^{2z}$$

The auxiliary equation is: $m^2 - 4m + 4 = 0$

$$m^2 - 2m - 2m + 4 = 0$$

$$m(m-2) - 2(m-2) = 0$$

$$m = 2, 2$$

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The complimentary function (CF): $(C_1+C_2 z)\, e^{2z}$

$$P.I = \frac{2e^{2z}}{D^2-4D+4}$$

$$= 2\left\{\frac{e^{2z}}{D^2-4D+4}\right\}$$

$D=a=2$

$$= 2\left\{\frac{e^{2z}}{0}\right\}$$

$$= 2\left\{\frac{e^{2z}}{2D-4}\right\}\cdot z$$

$$= 2\left\{\frac{e^{2z}}{2}\right\}\cdot z^2$$

$$P.I = e^{2z}\cdot z^2$$

$$P.I = e^{2(\log(2x+1))}\cdot \log(2x+1)^2$$

$$= e^{\log(2x+1)^2}\cdot \log(2x+1)^2$$

$$= (2x+1)^2\cdot \log(2x+1)^2$$

∴ The general solution is:

$$y = (C_1+C_2 z)\, e^{2\log(2x+1)} + (2x+1)^2\cdot \log(2x+1)^2$$

$$\boxed{y = C_1+C_2(\log(2x+1))\cdot(2x+1)^2 + (2x+1)^2\cdot \log(2x+1)^2}$$

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7. Solve: $(1+x)^2 y'' + (1+x)y' + y = \sin\{\log(1+x)^2\}$

Sol: Given, $(1+x)^2 y'' + (1+x)y' + y = \sin\{\log(1+x)^2\}$

$$(1+x)^2 y'' + (1+x)y' + y = \sin\{2\log(1+x)\} \longrightarrow (1)$$

Put $z = \log(1+x) \Rightarrow 1+x = e^z$

$$(1+x)\frac{dy}{dx} = Dy$$

$$(1+x)^2 \frac{d^2y}{dx^2} = D(D-1)y$$

$$(1) \Rightarrow \{D(D-1) + D + 1\} y = \sin\{2z\}$$

$$\{D^2 - D + D + 1\} y = \sin\{2z\}$$

$$\{D^2 + 1\} y = \sin(2z)$$

The auxiliary equation is: $m^2 + 1 = 0$

$$m^2 = -1$$

$$m = \pm\sqrt{-1}$$

$$\boxed{m = \pm i}$$

The complimentary function $\boxed{(CF): C_1 \cos z + C_2 \sin z}$

$$P.I = \frac{\sin 2z}{D^2+1}$$

Put $D^2 = -a^2 = -4$

$$P.I = \frac{1}{-3}\sin 2z$$

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∴ The General solution is:

$$y = C_1 \cos z + C_2 \sin z - \frac{\sin 2z}{3}$$

$$\boxed{y = C_1 \cos\{\log(1+x)\} + C_2 \sin\{\log(1+x)\} - \frac{\sin 2\{\log(1+x)\}}{3}}$$

8. Solve: $(2x+3)^2 y'' - (2x+3) y' - 12y = 6x$

Sol: Given $(2x+3)^2 y'' - (2x+3)y' - 12y = 6x$

Put $z = \log(2x+3) \Rightarrow 2x+3 = e^z$

$$(2x+3)\frac{dy}{dx} = 2Dy$$

$$(2x+3)^2 \frac{d^2y}{dx^2} = 4D(D-1)y$$

$$\{4D(D-1) - 2D - 12\}y = 6x$$

$$\{4D^2 - 4D - 2D - 12\}y = 6x$$

$$\{4D^2 - 6D - 12\}y = 6\left\{\frac{e^z - 3}{2}\right\}$$

∴ The auxiliary equation is: $4m^2 - 6m - 12 = 0$

$$m = \frac{6 \pm \sqrt{36 - 4 \cdot 4 \cdot (-12)}}{2 \cdot 4}$$

$$m = \frac{6 \pm \sqrt{36 + 192}}{8} = \frac{6 \pm \sqrt{228}}{8} = \frac{6 \pm 2\sqrt{57}}{8}$$

$$\boxed{m = \frac{3 \pm \sqrt{57}}{4}}$$

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∴ Complimentary function (CF): $C_1 e^{\left(\frac{3+\sqrt{57}}{4}\right)x} + C_2 e^{\left(\frac{3-\sqrt{57}}{4}\right)x}$

$$P.I = \frac{\frac{6e^z}{2} - \frac{18}{2}}{4D^2 - 6D - 12}$$

$$P.I = \frac{3e^z}{4D^2 - 6D - 12} - \frac{9 \cdot e^0}{4D^2 - 6D - 12}$$

$D = a = 1$ $\qquad$ $D = a = 0$

$$= 3\left\{\frac{e^z}{-14}\right\} - \frac{9}{-12}$$

$$P.I = -\frac{3e^z}{14} + \frac{9}{12}$$

∴ General solution is:

$$y = C_1 e^{\left(\frac{3+\sqrt{57}}{4}\right)\log(2x+3)} + C_2 e^{\left(\frac{3-\sqrt{57}}{4}\right)\log(2x+3)} + \frac{3}{4}\log(2x+1) - \frac{3e}{14}$$

9. Solve: $(2x-1)^2 y'' + (2x-1)y' - 2y = 8x^2 - 2x + 3$

Sol: Given, $(2x-1)^2 y'' + (2x-1)y' - 2y = 8x^2 - 2x + 3$

Put $z = \log(2x-1) \Rightarrow 2x - 1 = e^z \Rightarrow x = \frac{e^z + 1}{2}$

$$(2x-1)\frac{dy}{dx} = 2Dy$$

$$(2x-1)^2 \frac{d^2y}{dx^2} = 2D(D-1)y$$

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$$\{2D(D-1)+2D-2\}y = 8\left\{\frac{e^z+1}{2}\right\}^2 - 2\left\{\frac{e^z+1}{2}\right\} + 3$$

$$\{2(D^2-D)+2D-2\}y = 8\left\{\frac{e^{2z}}{4} + \frac{1}{4} + 2\cdot\frac{e^z}{2}\cdot\frac{1}{2}\right\} - e^z - 1 + 3$$

$$\{2D^2-2\}y = 2e^{2z} + 2 + 4e^z - e^z - 1 + 3$$

$$\{2D^2-2\}y = 5e^z + 4$$

The auxiliary equation is: $2m^2-2=0 \Rightarrow 2m^2=2$

$m^2=1$

$\boxed{m=\pm 1}$

∴ Complimentary function $\boxed{(CF): C_1e^z + C_2e^{-z}}$

$$PI = \frac{5e^z}{2D^2-2} + \frac{4\cdot e^0}{2D^2-2}$$

Put $D=a=1$ $\quad D=a=0$

$$= \frac{5e^z}{0} + \frac{4\cdot e^0}{-2}$$

$$= \frac{5e^z}{4D}\cdot z + \frac{4\cdot e^0}{-2D}$$

$D=a=1$

$$= \frac{5}{4}\cdot e^z\cdot z - 2$$

$$= \frac{5}{4}\left\{\log(2x-1)\cdot e^{\log(2x-1)}\right\} - 2$$

$$= \frac{5}{4}\log(2x-1)\cdot(2x-1) - 2$$

∴ General solution is $\boxed{y = C_1(2x-1) + C_2(2x-1)^{-1} + \frac{5}{4}\log(2x-1)(2x-1) - 2}$

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Method of variation of parameters:

Consider a differential equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

where R is a function of x.

Step-1: Find complimentary function (CF)

i.e., $CF = C_1u + C_2v$

Step-2: Assume the solution in the form:

$$\boxed{y = Au + Bv}$$

where $A = -\int \frac{vR}{W}\,dx$

$$B = \int \frac{uR}{W}\,dx$$

$$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix}$$

Step-3: Substitute these values in step-2 to get solution

Q. By the method of variation of parameters, solve: $y'' + a^2y = \sec(ax)$

Sol: Given, $y'' + a^2y = \sec(ax)$

$$(D^2 + a^2)y = \sec(ax)$$

The auxiliary equation is: $m^2 + a^2 = 0$

$$m^2 = -a^2$$

$$m = \pm\sqrt{-a^2}$$

$$\boxed{m = \pm ia}$$

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∴ Complimentary function (CF): $C_1 \cos ax + C_2 \sin ax$

$$CF = C_1 u + C_2 V$$

$$\therefore u = \cos ax, \; V = \sin ax$$

$$W = \begin{vmatrix} u & V \\ u_1 & V_1 \end{vmatrix}$$

$$W = \begin{vmatrix} \cos ax & \sin ax \\ -a \sin ax & a \cos ax \end{vmatrix}$$

$$W = a \cos^2 ax + a \sin^2 ax$$

$$W = a \left(\cos^2 ax + \sin^2 ax \right)$$

$$W = a$$

$$A = -\int \frac{VR}{W} \, dx$$

$$A = -\int \frac{\sin ax \cdot \sec ax}{a} \, dx$$

$$A = -\int \frac{\tan ax}{a} \cdot dx$$

$$A = -\frac{1}{a^2} \log \sec ax + C_1$$

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$$B = \int \frac{uR}{W}\, dx$$

$$B = \int \frac{\tan ax \cdot \sec ax}{a} \cdot dx$$

$$B = \int \frac{\tan x}{a} \cdot dx$$

$$B = \frac{1}{a^2} \log \sec ax + C_2$$

$$B = \int \frac{dx}{a} \Rightarrow \boxed{B = \frac{x}{a} + C_2}$$

∴ The General solution is : $y = Au + Bv$

$$y = \left(\frac{-1}{a^2} \log \sec ax + C_1\right) \cos ax + \left(\frac{x}{a} + C_2\right) \sin ax$$

$$\boxed{y = C_1 \cos ax + C_2 \sin ax - \frac{1}{a^2} \log \sec ax \cos ax + \frac{x}{a} \sin ax}$$

Use the method of variation of parameters to solve : $y'' - 2y' + y = e^x \log x$

Sol: Given, $y'' - 2y' + y = e^x \log x$

$$(D^2 - 2D + 1)y = e^x \log x$$

Its auxiliary equation is : $m^2 - 2m + 1 = 0$

$$m^2 - 1m - 1m + 1 = 0$$

$$m(m-1) - 1(m-1) = 0$$

$$\boxed{m = 1, 1}$$

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∴ Complimentary function : $\boxed{(C_1 + C_2 x)e^x}$

$$\boxed{CF: C_1 e^x + C_2 x e^x}$$

$$\therefore u = e^x, \quad V = x \cdot e^x$$

$$W = \begin{vmatrix} u & V \\ u_1 & V_1 \end{vmatrix} = \begin{vmatrix} e^x & x \cdot e^x \\ e^x & x e^x + e^x \end{vmatrix}$$

$$W = e^x\left(xe^x + e^x\right) - \left(xe^x\right)e^x$$

$$= x e^{2x} + e^{2x} - x e^{2x}$$

$$\boxed{W = e^{2x}}$$

$$A = -\int \frac{V R}{W} \, dx$$

$$A = -\int \frac{x \cdot e^x \cdot e^x \log x}{e^{2x}} \, dx$$

$$A = -\int \frac{x \cdot e^{2x} \log x}{e^{2x}} \, dx$$

$$A = -\int x \cdot \log x \, dx$$

$$A = -\int \log x \cdot x \, dx$$

$$A = -\left(\log x \cdot \frac{x^2}{2} - \int \log x \cdot \frac{x^2}{2} \cdot \frac{1}{x}\right) dx$$

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$$A = -\frac{x^2}{2}\log x + \frac{1}{2}\int x\,dx$$

$$\boxed{A = -\frac{x^2}{2}\log x + \frac{x^2}{4} + C_1}$$

$$B = \int \frac{uR}{W}\,dx$$

$$B = \int \frac{e^x \cdot e^x \log x}{e^{2x}}\,dx$$

$$B = \int \frac{e^{2x} \cdot \log x}{e^{2x}}\,dx$$

$$\boxed{B = x\log x - x + C_2}$$

∴ The general solution is: $\boxed{y = Au + Bv}$

$$y = \left(-\frac{x^2}{2}\log x + \frac{x^2}{4} + C_1\right)e^x + \left(x\log x - x + C_2\right)xe^x$$

$$\boxed{y = C_1 e^x + C_2 x e^x - \frac{x^2}{2}\log x\, e^x + \frac{x^2}{4}e^x + x^2\log x \cdot e^x - x^2 e^x}$$

3. Solve: $y'' - 6y' + 9y = \dfrac{e^{3x}}{x^2}$

Sol. Given, $y'' - 6y' + 9y = \dfrac{e^{3x}}{x^2}$

$$(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$$

The auxiliary equation is: $m^2 - 6m + 9 = 0$

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$m^2 - 3m - 3m + 9 = 0$

$m(m-3) - 3(m-3) = 0$

$\boxed{m = 3, 3}$

The complimentary equation $\boxed{(CF): (C_1 + C_2 x)e^{3x}}$

$$\boxed{CF: C_1 e^{3x} + C_2 x \cdot e^{3x}}$$

$$u = e^{3x}, \quad V = x \cdot e^{3x}$$

$$W = \begin{vmatrix} u & V \\ u_1 & V_1 \end{vmatrix} = \begin{vmatrix} e^{3x} & x \cdot e^{3x} \\ e^{3x} \cdot 3 & x \cdot e^{3x} \cdot 3 + e^{3x} \end{vmatrix}$$

$$W = e^{3x}\left(3x e^{3x} + e^{3x} - 3x \cdot e^{6x}\right)$$

$$= 3x e^{6x} + e^{6x} - 3x e^{6x}$$

$$\boxed{W = e^{6x}}$$

$$A = -\int \frac{VR}{W}\, dx$$

$$A = -\int \frac{x e^{3x} \cdot \frac{e^{3x}}{x}}{e^{6x}}\, dx$$

$$A = -\int \frac{e^{6x}}{x\, e^{6x}}\, dx$$

$$\boxed{A = -\log x + C_1}$$

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$$B = \int \frac{uR}{W}\, dx$$

$$B = \int \frac{e^{3x} \cdot \frac{e^{3x}}{x^2}}{e^{6x}}\, dx$$

$$B = \int \frac{e^{6x}}{x^2 \cdot e^{6x}}\, dx$$

$$B = \int x^{-2}\, dx$$

$$B = \frac{x^{-1}}{-1}$$

$$\boxed{B = \frac{-1}{x} + C_2}$$

∴ The general solution is: $y = Au + Bv$

$$y = (\log x + C_1)e^{3x} + \left(\frac{-1}{x} + C_2\right) x e^{3x}$$

$$y = C_1 e^{3x} + C_2 \cdot x e^{3x} - e^{3x} \log x - \frac{1}{x} \cdot x e^{3x}$$

$$\boxed{y = (C_1 + C_2 x)e^{3x} - e^{3x} \log x - e^{3x}}$$

4. Solve: $y'' + y = \frac{1}{1 + \sin x}$

Sol: Given, $y'' + y = \frac{1}{1 + \sin x}$

$$(D^2 + 1)y = \frac{1}{1 + \sin x}$$

The auxiliary equation is $m^2 + 1 = 0$

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$$m^2 = -1$$

$$m = \pm\sqrt{-1}$$

$$\boxed{m = \pm i}$$

∴ The ~~answer~~ complimentary function (CF): $C_1 \cos x + C_2 \sin x$

$$\therefore \quad u = \cos x, \; V = \sin x$$

$$W = \begin{vmatrix} u & V \\ u_1 & V_1 \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x$$

$$\boxed{W = 1}$$

$$A = -\int \frac{VR}{W}$$

$$= -\int \sin x \times \frac{1}{1+\sin x} \, dx$$

$$= -\int \frac{\sin x \times 1-\sin x}{(1+\sin x)(1-\sin x)} \, dx$$

$$= -\int \frac{\sin x - \sin^2 x}{\cos^2 x} \, dx$$

$$= -\int (\tan x \cdot \sec x - \tan^2 x) \, dx$$

$$= -\int \{\tan x \cdot \sec x - (\sec^2 x - 1)\} \, dx$$

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$$A = -\sec x + \tan x - x + c_1$$

$$B = \int \frac{uR}{W}$$

$$B = \int \frac{\cos x \times \frac{1}{1+\sin x}}{1}$$

$$B = \int \frac{\cos x}{1+\sin x}$$

$$B = \log(1+\sin x) + c_2$$

∴ The general solution is : $y = Au + Bv$

$$y = \left(\sec x + \tan x - x + c_1\right)\cos x + \left(\log(1+\sin x) + c_2\right)\sin x$$

$$y = c_1\cos x + c_2 \sin x - 1 + \sin x - x\cos x + \log(1+\sin x)\cdot\sin x$$

5. Solve: $(D^2+3D+2)y = e^{e^x}$

Sol: Given, $(D^2+3D+2)y = e^{e^x}$

The auxiliary equation is $m^2+3m+2=0$

$$m^2+2m+1m+2=0$$

$$m(m+2)+1(m+2)=0$$

$$m = -1, -2$$

∴ The complimentary function (CF): $c_1e^{-x} + c_2e^{-2x}$

$u = e^{-x}$, $V = e^{-2x}$

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$$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}$$

$$W = e^{-x}\left(-2e^{-2x}\right) - e^{-2x}\left(-e^{-x}\right)$$

$$= -2e^{-3x} + e^{-3x}$$

$$\boxed{W = -e^{-3x}}$$

$$A = -\int \frac{v R}{W}$$

$$= +\int \frac{e^{-2x} \times e^{e^x}}{-e^{-3x}} \, dx$$

$$A = \int e^x \cdot e^{e^x} \, dx$$

Put $e^x = t$

$e^x dx = dt$

$$A = \int e^t \, dt$$

$$A = e^t + c_1$$

$$\boxed{A = e^{e^x} + c_1}$$

$$B = \int \frac{u R}{W}$$

$$B = \int \frac{e^{-x} \times e^{e^x}}{-e^{-3x}}$$

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$$B = -\int e^{2x} \times e^{e^x}\, dx$$

$$B = -\int e^x \times e^x \times e^{e^x}\, dx$$

Put $e^x = t$

$$e^x dx = dt$$

$$B = -\int e^t \cdot t\, dt$$

$$B = -\int t \cdot e^t\, dt$$

$$B = -\int t \cdot e^t - \int e^t\, dt$$

$$= -\int t\, e^t - e^t$$

$$= -te^t + e^t$$

$$\boxed{B = -e^x \cdot e^{e^x} + e^{e^x} + C_2}$$

∴ General solution is: $y = Au + Bv$

$$y = \left(e^{e^x} + C_1\right)e^{-x} + \left(-e^x \cdot e^{e^x} + e^{e^x} + C_2\right)e^{-2x}$$

$$y = C_1 e^{-x} + C_2 e^{-2x} + e^{-x} \cdot e^{e^x} - e^x \cdot e^{e^x} e^{-2x} + e^{e^x} \cdot e^{-2x}$$

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6. Solve: $(y''+y) = x \sin x$

Sol: Given, $(y''+y) = x \sin x$

$(D^2+1)y = x \sin x$

The auxiliary equation is: $m^2+1=0$

$$\boxed{m = \pm i}$$

$\therefore$ Complimentary function $\boxed{(CF): C_1 \cos x + C_2 \sin x}$

$$u = \cos x, \quad v = \sin x$$

$$w = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$w = \cos^2 x + \sin^2 x$$

$$w = 1$$

$$A = -\int \frac{vR}{w} = -\int \frac{\sin x \times x \sin x}{1} \, dx$$

$$A = -\int x \cdot \sin^2 x$$

$$A = -\int x \cdot \left(\frac{1-\cos 2x}{2}\right) dx$$

$$A = -\int \frac{x}{2} - \frac{x \cos 2x}{2} \, dx$$

$$A = -\frac{1}{2}\left[\int x \, dx\right] + \frac{1}{2}\left[\int x \cos 2x\right] dx$$

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$$A = \frac{-1}{2} \cdot \frac{x^2}{2} + \frac{1}{2}\left[x \cdot \left(\frac{\sin 2x}{2}\right) - 1\left(-\frac{\cos 2x}{4}\right)\right]$$

$$\boxed{A = \frac{-x^2}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + C_1}$$

$$B = \int \frac{uR}{W} \, dx$$

$$= \int \frac{\cos x \times x \sin x}{1} \, dx$$

$$B = \int x \cdot \cos x \sin x \, dx$$

$$B = \int x \cdot \frac{\sin 2x}{2} \, dx$$

$$\left[\because \sin 2x = 2 \sin x \cos x \right.$$
$$\left. \sin x \cos x = \frac{\sin 2x}{2}\right]$$

$$B = \frac{1}{2} \int x \cdot \sin 2x \, dx$$

$$B = \frac{1}{2}\left[x \cdot \left(\frac{-\cos 2x}{2}\right) - 1\left(\frac{-\sin 2x}{4}\right)\right] + C_2$$

$$B = \frac{-x \cos 2x}{4} + \frac{\sin 2x}{8} + C_2$$

∴ The General solution is : $y = Au + Bv$

$$y = \left(\frac{-x^2}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + C_1\right)\cos x + \left(\frac{-x \cos 2x}{4} + \frac{\sin 2x}{8} + C_2\right)\sin x$$

$$y = C_1 \cos x + C_2 \sin x - \frac{x^2 \cos x}{4} + \frac{x \sin 2x \cos x}{4} + \frac{\cos 2x \cdot \cos x}{8}$$

$$- \frac{x \cos 2x \sin x}{4} + \frac{\sin 2x \cdot \sin x}{8}$$

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7. Solve: $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

Sol: Given, $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$

$$(D^2-1)y = \frac{2}{1+e^x}$$

The auxiliary equation is: $m^2-1=0$

$$m^2 = 1$$

$$\boxed{m = \pm 1}$$

$\therefore$ The complimentary function $\boxed{(CF): C_1e^x + C_2e^{-x}}$

$$u = e^x, \quad v = e^{-x}$$

$$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$$

$$W = e^x \cdot -e^{-x} - \left(e^{-x} \cdot e^x\right)$$

$$W = -e^0 - e^0$$

$$\boxed{W = -2}$$

$$A = -\int \frac{vR}{W}$$

$$A = -\int \frac{e^{-x} \times \frac{2}{1+e^x}}{-2} \, dx$$

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$$A = \int e^{-x} \times \frac{2}{1+e^{x}} \times \frac{1}{2}$$

$$A = \int \frac{e^{-x}}{1+e^{x}} \, dx$$

Add & substract e^{x}

$$A = \int \frac{(1+e^{x}) - e^{x}}{e^{x}(1+e^{x})} \, dx$$

$$A = \int \left(\frac{1}{e^{x}} - \frac{1}{1+e^{x}} \right) dx$$

$$A = \int \left(e^{-x} - \frac{(1+e^{x}) - e^{x}}{1+e^{x}} \right) dx$$

$$A = \int \left(e^{-x} - 1 + \frac{e^{x}}{1+e^{x}} \right) dx$$

$$A = \left\{ -e^{-x} - x + \log(1+e^{x}) \right\} + C_1$$

$$B = \int \frac{uR}{w} \, dx$$

$$B = \int \frac{e^{x} \times \frac{2}{1+e^{x}}}{-2} \, dx$$

$$B = -\int \frac{e^{x}}{1+e^{x}} \, dx$$

$$\boxed{B = -\log(1+e^{x}) + C_2}$$

General solution is: $y = C_1 e^{x} + C_2 e^{-x} + \left(-e^{-x} - x + \log(1+e^{x})\right) e^{x} + \left(-\log(1+e^{x})\right) \cdot e^{-x}$

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8. Solve: $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{1}{1+e^{-x}}$

Sol: Given, $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{1}{1+e^{-x}}$

$$(D^2 - 3D + 2)y = \frac{1}{1+e^{-x}}$$

The auxiliary equation is: $m^2 - 3m + 2 = 0$

$$m^2 - 2m - 1m + 2 = 0$$

$$m(m-2) - 1(m-2) = 0$$

$$\boxed{m = 2, 1}$$

The complimentary function : $\boxed{CF : C_1e^x + C_2e^{2x}}$

$$\boxed{u = e^x, V = e^{2x}}$$

$$W = \begin{vmatrix} u & V \\ u_1 & V_1 \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix}$$

$$W = 2 \cdot e^x \cdot e^{2x} - e^{2x} \cdot e^x$$

$$\boxed{W = e^{3x}}$$

$$A = -\int \frac{VR}{W}\, dx$$

$$= -\int \frac{e^{2x} \times \frac{1}{1+e^{-x}}}{e^{3x}}\, dx$$

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$$A = -\int \frac{e^{2x-3x}}{1+e^{-x}}\,dx$$

$$A = -\int \frac{e^{-x}}{1+e^{-x}}\,dx$$

$$A = -1 \times \frac{-1}{-1}\int \frac{e^{-x}}{1+e^{-x}}\,dx$$

$$A = \int \frac{-e^{-x}}{1+e^{-x}}\,dx$$

$$\boxed{A = \log\left(1+e^{-x}\right) + C_1}$$

$$B = \int \frac{uR}{W}\,dx$$

$$B = \int \frac{e^{x} \times \frac{1}{1+e^{-x}}}{e^{2x}}$$

$$B = \int \frac{e^{-2x}}{1+e^{-x}}\,dx$$

$$B = \int \frac{e^{-x}\cdot e^{-x}}{1+e^{-x}}\,dx$$

$$B = \int \frac{1}{e^{2x}\left(1+e^{-x}\right)}\,dx$$

$$1+e^{-x} = t$$

$$-e^{-x}dx = dt$$

$$\boxed{e^{-x} = t-1}$$

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$$= \int \frac{e^{-x} \cdot e^{-x}}{(1+e^{-x})} \, dx$$

$$= \int \frac{(t-1)x - dt}{t}$$

$$= + \int \frac{1-t}{t} \, dt$$

$$= + \int \frac{1}{t} - 1 \, dt$$

$$= + (\log t - t) + c_1$$

$$= \log t - t + c_1$$

$$= \log(1+e^{-x}) - (1+e^{-x}) + c_2$$

$$\therefore S \text{ is } \cdot \quad y = Au + Bv$$

$$y = (\log(1+e^{-x})$$

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Applications

1. A particle undergoes vibrations according to $\frac{d^2x}{dt^2} + 25x = 21\cos 2t$. If the particle starts from rest at $t=0$, find the displacement at any time t.

Sol: Given, $\frac{d^2x}{dt^2} + 25x = 21\cos 2t$

$$(D^2+25)x = 21\cos 2t$$

The auxiliary equation is, $m^2+25=0$

$$m^2 = -25$$

$$\boxed{m = \pm 5i}$$

$\therefore$ Complimentary function $\boxed{(CF): C_1\cos 5t + C_2 \sin 5t}$

$$PI = 21\left\{\frac{\cos 2t}{D^2+25}\right\}$$

Put $D^2 = -a^2 = -4$

$$= 21\left\{\frac{\cos 2t}{21}\right\}$$

$$PI = \cos 2t$$

$\therefore$ The general solution is: $x = CF + PI$

$$x = C_1\cos 5t + C_2 \sin 5t + \cos 2t$$

When $t=0$, $x=0$

$$0 = C_1 + 1 \Rightarrow \boxed{C_1 = -1}$$

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$$x' = c_1(-5\sin 5t) + c_2(5\cos 5t) - 2\sin 2t$$

When $t = 0$, $x' = 0$

$$0 = 5c_2 \Rightarrow \boxed{c_2 = 0}$$

$$\therefore x = -\cos 5t + \cos 2t$$

2. Solve the problem of forced vibrations of spring governed by the differential equation: $m\frac{d^2y}{dt^2} + Ky = f(t)$ in the case where the forcing function $f(t) = a\sin\omega t$.

Take $y(0) = y_0$, $y'(0) = y_1$

Sol: Given, $m\frac{d^2y}{dt^2} + Ky = f(t)$

$$m\frac{d^2y}{dt^2} + Ky = a\sin\omega t$$

Divide equation by m

$$\frac{d^2y}{dt^2} + \frac{K}{m}y = \frac{a}{m}\sin\omega t$$

$$(D^2 + \lambda^2)y = \frac{A}{m}\sin\omega t \qquad \text{where } \lambda^2 = \frac{K}{m}$$

The auxiliary equation is: $D^2 + \lambda^2 = 0$

$$m^2 + \lambda^2 = 0$$

$$m^2 = -\lambda^2$$

$$\boxed{m = \pm i\lambda}$$

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CF: $C_1 \cos\lambda t + C_2 \sin\lambda t$

$$PI = \frac{\frac{a}{m}\sin\omega t}{D^2+\lambda^2}$$

$$= \frac{a}{m}\left[\frac{\sin\omega t}{D^2+\lambda^2}\right]$$

Put $D^2 = -a^2 = -\omega^2$

$$= \frac{a}{m}\left[\frac{\sin\omega t}{\lambda^2-\omega^2}\right]$$

$$PI = \frac{\frac{A}{m}\sin\omega t}{\lambda^2-\omega^2}$$

General solution is: $y = CF + PI$

$$y = C_1\cos\lambda t + C_2\sin\lambda t + \frac{A}{m}\frac{\sin\omega t}{\lambda^2-\omega^2}$$

When $t=0$, $y=y_0$

$$y_0 = C_1$$

$$y' = -C_1\cdot\lambda\sin\lambda t + C_2\lambda\cos\lambda t + \frac{A}{m}\frac{\omega\cos\omega t}{(\lambda^2-\omega^2)}$$

When $t=0$, $y'(0)=y_1$

$$y_1 = C_2\lambda + \frac{A\omega}{m(\lambda^2-\omega^2)} \Rightarrow C_2 = \frac{1}{\lambda}\left\{y_1 - \frac{A\omega}{m(\lambda^2-\omega^2)}\right\}$$

Solution is: $y = y_0\cos\lambda t + \frac{1}{\lambda}\left(y_1 - \frac{A\omega}{m(\lambda^2-\omega^2)}\right) + \frac{A}{m}\frac{\sin\omega t}{\lambda^2-\omega^2}$

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Note: The differential equation of the LCR circuit containing the EMF E is given by: $\boxed{L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=E}$

1. An uncharged condenser of capacity C is charged by applying an EMF $E\sin\left(\frac{t}{\sqrt{LC}}\right)$ through the self inductance L and negligible resistance R. Prove that the charge on one of the plates is:

$$\frac{EC}{2}\left\{\sin\left(\frac{t}{\sqrt{LC}}\right)-\frac{t}{\sqrt{LC}}\cos\left(\frac{t}{\sqrt{LC}}\right)\right\}$$

Sol: The differential equation governing LCR circuit is given by:

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=E\sin\left(\frac{t}{\sqrt{LC}}\right)$$

Since, the resistance is negligible, the differential equation reduces to:

$$L\frac{d^2q}{dt^2}+\frac{q}{C}=E\sin\left(\frac{t}{\sqrt{LC}}\right)$$

$$\frac{d^2q}{dt^2}+\frac{q}{LC}=\frac{E}{L}\sin\left(\frac{t}{\sqrt{LC}}\right)$$

$$\left(D^2+\frac{1}{LC}\right)q=\frac{E}{L}\sin\left(\frac{t}{\sqrt{LC}}\right)$$

The auxiliary equation is: $m^2+\frac{1}{LC}=0$

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$$m^2 = \frac{-1}{LC}$$

$$m = \pm\sqrt{-1}\sqrt{\frac{1}{LC}}$$

$$\boxed{m = \pm i \times \frac{1}{\sqrt{LC}}}$$

$$\therefore \boxed{CF : C_1 \cos\left(\frac{1}{\sqrt{LC}}\right)t + C_2 \sin\left(\frac{1}{\sqrt{LC}}\right)t}$$

$$PI = \frac{\frac{E}{L}\sin\left(\frac{t}{\sqrt{LC}}\right)}{D^2 + \frac{1}{LC}}$$

Put $D^2 = -a^2 = -\left(\frac{1}{\sqrt{LC}}\right)^2 = \frac{-1}{LC}$

$$PI = \frac{\frac{E}{L}\sin\left(\frac{t}{\sqrt{LC}}\right)}{2D} \cdot t$$

$$PI = \frac{E}{2L}\left\{\frac{-\cos\left(\frac{t}{\sqrt{LC}}\right)}{\frac{1}{\sqrt{LC}}}\right\} \cdot t$$

$$PI = \frac{E}{2L} \times \sqrt{LC} \cdot \left(-\cos\left(\frac{t}{\sqrt{LC}}\right)\right) \cdot t$$

$$PI = -\frac{E\sqrt{C}}{2\sqrt{L}} \cdot t\cos\left(\frac{t}{\sqrt{LC}}\right)$$

$\therefore$ General solution is $q = C_1 \cos\left(\frac{1}{\sqrt{LC}}\right)t + C_2 \sin\left(\frac{1}{\sqrt{LC}}\right)t$

$$+ \left(\frac{-E\sqrt{C}}{2\sqrt{L}} \cdot t\cos\left(\frac{t}{\sqrt{LC}}\right)\right)$$

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When $t=0, q=0$

$$0 = C_1 - \frac{E\sqrt{C}}{2\sqrt{L}}\, t$$

$$C_1 = \frac{E\sqrt{C}}{2\sqrt{L}}\, t \quad \Rightarrow \boxed{C_1 = 0}$$

When $t=0$, $i = \frac{dq}{dt} = 0$

$$\frac{dq}{dt} = C_1\left(-\sin\left(\frac{1}{\sqrt{LC}}\right)\right)t \cdot \frac{1}{\sqrt{LC}} + C_2 \cos\left(\frac{1}{\sqrt{LC}}\right)\cdot t \cdot \frac{1}{\sqrt{LC}}$$

$$\frac{-E\sqrt{C}}{2\sqrt{L}}\, t\left(\frac{-\sin t}{\sqrt{LC}}\right)\cdot\frac{1}{\sqrt{LC}}$$

$$\frac{dq}{dt} = \frac{d}{dt}\left[\frac{E\sqrt{C}}{2\sqrt{L}}\, t \cos\left(\frac{t}{\sqrt{LC}}\right)\right]$$

$$\frac{dq}{dt} = -\frac{E\sqrt{C}}{2\sqrt{L}}\left\{\left[t\cdot\left(-\sin\left(\frac{t}{\sqrt{LC}}\right)\cdot\frac{1}{\sqrt{LC}}\right)\right] + \cos\left(\frac{t}{\sqrt{LC}}\right)\cdot 1\right\}$$

When $t=0$, $\frac{dq}{dt} = 0$

$$\therefore 0 = \frac{C_2}{\sqrt{LC}} - \frac{E\sqrt{C}}{2\sqrt{L}}$$

$$C_2 = \sqrt{LC}\left[\frac{E\sqrt{C}}{2\sqrt{L}}\right]$$

$$\boxed{C_2 = \frac{EC}{2}}$$

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G.S is $q = \frac{EC}{2}\sin\left(\frac{t}{\sqrt{LC}}\right) - \frac{E\sqrt{C}}{2\sqrt{L}} \cdot t\cos\left(\frac{t}{\sqrt{LC}}\right)$

$$q = \frac{EC}{2}\left\{\sin\left(\frac{t}{\sqrt{LC}} - \frac{t}{\sqrt{LC}}\cos\left(\frac{t}{\sqrt{LC}}\right)\right\}\right.$$

The current i, the charge q in a series circuit containing the resistance R, the inductance L and capacitance C, EMF E satisfy the differential equation: $L\frac{di}{dt} + \frac{q}{C} = E$. Assuming that resistance is negligible. Express q and i in terms of t. Given that i and q are both zero initially.

Sol: Given, $L\frac{di}{dt} + \frac{q}{C} = E$

$$\boxed{i = \frac{dq}{dt}}$$

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = E$$

$$\frac{d^2q}{dt^2} + \frac{q}{LC} = E$$

$$\left(D^2 + \frac{1}{LC}\right)q = \frac{E}{L}$$

The auxiliary equation is: $m^2 + \frac{1}{LC} = 0$

$$m^2 = -\frac{1}{LC}$$

$$\boxed{m = \pm\frac{i}{\sqrt{LC}}}$$

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$\therefore$ $\boxed{CF: C_1 \cos\frac{t}{\sqrt{LC}} + C_2 \sin\frac{t}{\sqrt{LC}}}$

$$PI = \frac{E/L}{D^2 + \frac{1}{LC}}$$

$$= \frac{E}{L}\left\{\frac{1 \cdot e^{0}}{D^2 + \frac{1}{LC}}\right\}$$

$$D = a = 0$$

$$= \frac{E}{L}\left\{\frac{1}{\frac{1}{LC}}\right\}$$

$$PI = \frac{E}{L} \times LC$$

$$\boxed{PI = EC}$$

$\therefore$ General solution is $\boxed{q = C_1 \cos\frac{t}{\sqrt{LC}} + C_2 \sin\frac{t}{\sqrt{LC}} + EC}$

When $t = 0,\ q = 0$

$$0 = C_1 + EC$$

$$\boxed{C_1 = -EC}$$

$$\frac{dq}{dt} = \frac{C_1}{\sqrt{LC}}\left(-\sin\frac{t}{\sqrt{LC}}\right) + \frac{C_2}{\sqrt{LC}} \cdot \cos\left(\frac{t}{\sqrt{LC}}\right).$$

$\boxed{C_2 = 0}$

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∴ General solution is :

$$q = -EC \cos\frac{t}{\sqrt{LC}} + EC$$

$$\boxed{q = EC\left(1 - \cos\frac{t}{\sqrt{LC}}\right)}$$

$$\frac{dq}{dt} = -\frac{EC}{\sqrt{LC}}\left(-\sin\frac{t}{\sqrt{LC}}\right)$$

$$\boxed{i = \frac{EC}{\sqrt{LC}}\left(\sin\frac{t}{\sqrt{LC}}\right)}$$

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Partial Differential Equations.

A differential eqn involving a dependent variable which is function of two or more independent variables and its partial derivatives is known as a partial differential equation.

Generally PDE is denoted by $f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \ldots\right) = 0$

The highest order derivative present in a PDE is the order of PDE and the positive integral power of highest order derivative is the degree of PDE

Ex:-

	Order	Degree
$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial y} + u = 0$	2	1
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u$	2	1
$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$	2	1
$\left(\frac{\partial u}{\partial x}\right)^{3/2} = \frac{\partial^2 u}{\partial y^2}$	2	2

SQB S

$\left(\frac{\partial u}{\partial x}\right)^3 = \left(\frac{\partial^2 u}{\partial x^2}\right)^2$

Solution of PDE is a relation b/w dependent & independent variable that satisfies given PDE. It can be an implicit or explicit function.

From a given relation b/w dependent and independent variables the process of finding a diff eqn is known as formation of DE. A PDE can be formed by eliminating arbitrary ① constants ② functions.

In formation of PDE by eliminating arbitrary constants if the number of independent variables is equal to no of arbitrary constants then we get first order PDE otherwise we get higher order PDE.

In formation of PDE by eliminating arbitrary function we get 1st order PDE if we have only one function otherwise we get higher order PDE

The following is the notations used in representing PDE and solutions.

$z = z(x,y)$ or $u = u(x,y)$

$\frac{\partial z}{\partial x} = p = z_x$

$\frac{\partial z}{\partial y} = q = z_y$

$\frac{\partial^2 z}{\partial x^2} = r = z_{xx}$

$\frac{\partial^2 z}{\partial x \partial y} = s = z_{xy} = z_{yx}$

$\frac{\partial^2 z}{\partial y^2} = t = z_{yy}$

In the following

1. $Z = (a+x^2)(b+y^2)$ try to remove a, b – arb const

$$\frac{\partial Z}{\partial x} = 2x\,(b+y^2)$$

$$\frac{\partial Z}{\partial y} = (a+x^2)\cdot 2y$$

Consider $\frac{\partial Z}{\partial x}\cdot\frac{\partial Z}{\partial y} = 4xy\,(a+x^2)(b+y^2)$

$= 4xyZ$

2. $Z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

Diff

$$2\cdot\frac{\partial Z}{\partial x} = \frac{2x}{a^2} + 0$$

multiply x on bs

$$x\cdot\frac{\partial Z}{\partial x} = \frac{x^2}{a^2}$$

$$2\cdot\frac{\partial Z}{\partial y} = 0 + \frac{2y}{b^2}$$

mul y on bs

$$y\frac{\partial Z}{\partial y} = \frac{y^2}{b^2}$$

Sub

$$2Z = x\cdot\frac{\partial Z}{\partial x} + y\cdot\frac{\partial Z}{\partial y}$$

3. $ax + by + z = 1$ → p → from pw paper

$$2ax + 0 + 2z\frac{\partial z}{\partial x} = 0 \Rightarrow ax = -zp$$

$$2ax = -2zp$$
$$ax = -zp \Rightarrow ax^2 = -xzp$$

$$0 + 2by + 2z\frac{\partial z}{\partial y} = 0 \Rightarrow by = -zq$$
$$by^2 = -yzq$$

$$\therefore ax^2 + by^2 + z^2 = 1$$
$$-xzp - yzq + z^2 = 1$$

4. $z = a \log(x^2 + y^2) + b.$

$$\frac{\partial z}{\partial x} = a \cdot \frac{2x}{x^2+y^2} + 0 \Rightarrow p = \frac{2x \cdot a}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = a \cdot \frac{2y}{x^2+y^2} + 0 \Rightarrow q = \frac{2y \cdot a}{x^2+y^2}$$

$$\frac{p}{q} = \frac{2x \cdot a}{2y \cdot a} \quad \therefore \quad \frac{p}{q} = \frac{x}{y}$$

5. $x^2 + y^2 = (z-c)^2 \tan^2\alpha$

Diff wrt x

$$2x + 0 = 2[z-c] \cdot \left(\frac{\partial z}{\partial x} - 0\right) \cdot \tan^2\alpha \quad \rightarrow (1)$$

Diff wrt y

$$0 + 2y = 2[z-c] \cdot \left[\frac{\partial z}{\partial y} - 0\right] \cdot \tan^2\alpha \quad \rightarrow (2)$$

$$\frac{(1)}{(2)} \quad \frac{2x}{2y} = \frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} \quad \frac{x}{y} = \frac{p}{q} \quad \therefore xq = yp$$

6. $z = a \log \frac{b(y-1)}{1-x}$

$z = a[\log b + \log(y-1) - \log(1-x)]$

Diff wrt x

$\frac{\partial z}{\partial x} = a\left[-\frac{1}{(1-x)}(-1)\right]$

$\frac{\partial z}{\partial x} = \frac{a}{(1-x)} = p$

Diff wrt y

$\frac{\partial z}{\partial y} = a\left[\frac{1}{y-1}\right] = \frac{a}{y-1} = q$

$\frac{p}{q} = \frac{y-1}{1-x}$

$(1-x)p = q(y-1)$

7. $z = ax + by + cxy$ ⇒ 3 arb can diff 3

① ②

$p = \frac{\partial z}{\partial x} = a + 0 + cy \cdot 1 = a + cy$, $r = \frac{\partial^2 z}{\partial x^2} = 0$

$q = \frac{\partial z}{\partial y} = 0 + b + cx = b + cx$, $t = \frac{\partial^2 z}{\partial y^2} = 0$

$s = \frac{\partial^2 z}{\partial x \partial y}$ ⊠ C

$p = a + sy$ $q = b + sx$

$a = (p - sy)$ $b = (q - sx)$

$\boxed{px + qy - sxy = 0}$

$z = (p - sy)x + (q - sx)y + sxy$

$(x-a)^2 + (y-b)^2 + z^2 = 1$

$2(x-a) + 0 + 2z \cdot \frac{\partial z}{\partial x} = 0$

$2(x-a) \cdot -2z \cdot p$

$(x-a) \cdot -z \cdot p$

$0 + 2(y-b) + 2z \frac{\partial z}{\partial y} = 0$

$\Rightarrow 2(y-b) \cdot -2z \frac{\partial z}{\partial y}$

$(y-b) \cdot -zq$

$\therefore (x-a)^2 + (y-b)^2 + z^2 = 1$

$z^2p^2 + z^2q^2 + z^2 = 1$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Diff wrt x

$\frac{2x}{a^2} + 0 + \frac{2z}{c^2} \cdot \frac{\partial z}{\partial x} = 0.$

$\frac{2x}{a^2} \cdot -\frac{2z}{c^2} \cdot \frac{\partial z}{\partial x}$

$\frac{x}{a^2} \cdot -\frac{z}{c^2} p$

$\frac{c^2}{a^2} \cdot \frac{-z}{x} \frac{\partial z}{\partial x}$ uvw rule $z \cdot \frac{1}{x} \cdot \frac{\partial z}{\partial x}$

$u \; v \; w$

Diff wrt x

$0 \cdot -\frac{z}{x} \cdot \frac{\partial^2 z}{\partial x^2} - \frac{-z}{x^2} \frac{\partial z}{\partial x} - \frac{1}{x}\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial x}\right)$

$\therefore -\frac{z}{x} \frac{\partial^2 z}{\partial x^2} + \frac{z}{x^2} \frac{\partial z}{\partial x} - \frac{1}{x}\left(\frac{\partial z}{\partial x}\right)^2 = 0$ //

Form the differential equation by eliminating arbitrary function from the following

only one func to only 1 derivative

$$z = f(x^2 - y^2)$$

$$\frac{\partial z}{\partial x} = f'(x^2-y^2) \cdot 2x = p$$

$$\frac{\partial z}{\partial y} = f'(x^2-y^2) \cdot -2y = q$$

$$\frac{p}{q} = \frac{2x}{-2y} = -py = xq$$

$$xq + yp = 0$$

$$z = f\left(\frac{xy}{z}\right)$$

diff wrt x, y is kept const & vice versa

$$\frac{\partial z}{\partial x} = f'\left(\frac{xy}{z}\right) \cdot \left(\frac{y}{z}\right)$$

$$\frac{\partial z}{\partial x} = f'\left(\frac{xy}{z}\right) \cdot \frac{y\left[z \cdot 1 - x \cdot \frac{\partial z}{\partial x}\right]}{z^2} = p$$

$$\frac{\partial z}{\partial y} = f'\left(\frac{xy}{z}\right) \cdot \frac{x\left[z \cdot 1 - y \cdot \frac{\partial z}{\partial y}\right]}{z^2} = q$$

dividing $\frac{p}{q}$

$$\frac{p}{q} = \frac{y(z - xp)}{x(z - yq)}$$

$$p \cdot x(z - yq) = q \cdot y(z - xp)$$

$$xzp - xpyq = yzq - xpyq$$

$$xzp = yzq$$

$$xp = yq$$

3. $z = y^2 + 2f\left[\frac{1}{x} + \log y\right]$

$z - y^2 = 2f\left[\frac{1}{x} + \log y\right]$

diff wrt x,

$$\frac{\partial z}{\partial x} - 0 = 2f'\left(\frac{1}{x} + \log y\right) \cdot \frac{-1}{x^2}$$

$$p - 0 = 2f'\left(\frac{1}{x} + \log y\right) \cdot \frac{-1}{x^2}.$$

diff wrt y;

$$\frac{\partial z}{\partial y} - 2y = 2f'\left(\frac{1}{x} + \log y\right) \cdot \left(0 + \frac{1}{y}\right).$$

$$q - 2y = 2f'\left(\frac{1}{x} + \log y\right) \cdot (1/y).$$

dividing: $\frac{p}{q - 2y} = \frac{-1/x^2}{1/y}$

$$= \frac{-y}{x^2}$$

$$px^2 = -qy + 2y^2$$

4. $lx + my + nz = \phi(x^2 + y^2 + z^2)$

Diff wrt x

$$lx\,1 + 0 + n \cdot \frac{\partial z}{\partial x} = \phi'(x^2 + y^2 + z^2) \cdot 2x + 0 + 2z \cdot \frac{\partial z}{\partial x}$$

Diff wrt y

$$0 + m \cdot 1 + n \cdot \frac{\partial z}{\partial y} = \phi'(x^2 + y^2 + z^2) \cdot 0 + 2y + 2z \cdot \frac{\partial z}{\partial y}$$

$$\frac{l + np}{m + nq} = \frac{2x + 2z \cdot p}{2y + 2zq}$$

5. $xyz = \phi(x+y+z)$.

$yz \cdot \frac{\partial x}{\partial x} = \phi'(x+y+z) \cdot \left(1 + \frac{\partial z}{\partial x}\right)$ $\quad \frac{yz \cdot p}{xz \cdot q} = \frac{1+p}{1+q}$

$xz \cdot \frac{\partial z}{\partial y} = \phi'(x+y+z) \cdot \left(1 + \frac{\partial z}{\partial y}\right)$

$\frac{yz/p}{xz \cdot q} = \frac{p}{q}$ $\quad \frac{y}{x} = \frac{x}{y}$

$x = y$

6. $z = f(x+ct) + g(x-ct)$

$p = \frac{\partial z}{\partial x} = f'(x+ct) + g'(x-ct)$

$r = \frac{\partial^2 z}{\partial x^2} = f''(x+ct) + g''(x-ct)$

$q = \frac{\partial z}{\partial t} = f'(x+ct) \cdot c + g'(x-ct) \cdot (-c)$

$t = \frac{\partial^2 z}{\partial t^2} = f''(x+ct) \cdot c^2 + g''(x-ct) \cdot (-c)^2$ $\quad -c \times -c = c^2$

$= c^2 \left[f''(x+ct) + g''(x-ct)\right]$

$t = c^2 r$

$\therefore t = c^2 r$ or $\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}$

7. $z = x\phi(y) + y\psi(x)$

$p = \frac{\partial z}{\partial x} = \phi(y) + y\psi'(x)$

$r = \frac{\partial^2 z}{\partial x^2} = 0 + y\psi''(x)$

$q = \frac{\partial z}{\partial y} = x\phi'(y) + \psi(x)$

$t = \frac{\partial^2 z}{\partial y^2} = x\phi''(y)$

$s = \frac{\partial^2 z}{\partial x \partial y} = \phi'(y) + \psi'(x)$

$px + qy = x\phi(y) + xy\psi'(x) + xy\phi'(y) + y\psi(x)$

$= x\phi(y) + y\psi(x) + xy[\psi'(x) + \phi'(y)]$

$px + qy = z + xys$

8. $z = \frac{1}{y}[f(x+ay) + g(x-ay)]$

$p = \frac{\partial z}{\partial x}$: $yz = f(x+ay) + g(x-ay)$

$y \cdot \frac{\partial z}{\partial x} = f'(x+ay) + g'(x-ay)$

$y \cdot \frac{\partial^2 z}{\partial x^2} = f''(x+ay) + g''(x-ay)$

$y \cdot \frac{\partial z}{\partial y} = f'(x+ay) \cdot a + g'(x-ay)(-a)$

$y \cdot \frac{\partial^2 z}{\partial y^2} = f''(x+ay) \cdot a^2 + g''(x-ay)(-a)^2$

$$yt + q + q = f''(x+ay)\,a^2 + g''(x-ay)(-a)^2$$

$$- a^2 [f''(x+ay) + g''(x-ay)]$$

$$yt + 2q = a^2 yr$$

$$- y \cdot \frac{\partial^2 z}{\partial y^2} + 2q = a^2 y \cdot \frac{\partial^2 z}{\partial x^2}$$

---

9. $u = f(y-2x) + g(y-3x)$

$$p = \frac{\partial u}{\partial x} = f'(y-2x).(-2) + g'(y-3x)(-3)$$

$$r = \frac{\partial^2 u}{\partial x^2} = f''(y-2x).(-2)^2 + g''(y-3x)(-3)^2$$

$$= 4f''(y-2x) + 9g''(y-3x) \longrightarrow (1)$$

$$= 4f'' + 9g''$$

$$q = \frac{\partial u}{\partial y} = f'(y-2x). + g'(y-3x).$$

$$t = \frac{\partial^2 u}{\partial y^2} = f''(y-2x) + g''(y-3x) \longrightarrow (2)$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = f''(y-2x).(-2) + g''(y-3x)(-3). \longrightarrow (3)$$

$$= -2f'' - 3g''$$

Consider $r = 4f'' + 9g''$

$s = -2f'' - 3g'' \quad \times 2$

$\Rightarrow r = 4f'' + 9g''$

$2s = -4f'' - 6g''$

$r + 2s = 3g''$

$$g'' = \frac{r+2s}{3}$$

$r = 4f'' + 9g''$

$3s = -6f'' - 9g''$

$r + 3s = -2f''$

$f'' = \frac{r+3s}{-2}$

$\therefore$ (2) $\Rightarrow$ $t = f'' + g''$

$t = \frac{r+3s}{-2} + \frac{r+2s}{3}$

$t = \frac{-3r - 9s + 2r + 4s}{6}$

$6t = -r - 5s$

$r + 6t + 5s = 0$

10. $z = f(2x+3y) + g(x-2y)$

p. $\frac{\partial z}{\partial x} = f'(2x+3y) \cdot 2 + g'(x-2y) \cdot 1$

r. $\frac{\partial^2 z}{\partial x^2} = f''(2x+3y) \cdot 4 + g''(x-2y) \cdot \Rightarrow 4f'' + g'' \rightarrow (1)$

q. $\frac{\partial z}{\partial y} = f'(2x+3y) \cdot 3 + g''(x-2y)(-2)$

t. $\frac{\partial^2 z}{\partial y^2} = f''(2x+3y) \cdot 9 + g''(x-2y) 4$

$9f''(2x+3y) + 4g''(x-2y)$

$9f'' + 4g'' \rightarrow (2)$

Solution of PDE by direct Integration method.

In this method, we solve a non-homogeneous PDE.

given a non homogeneous PDE we integrate it to get the solution z using the basic rules of Integration

While Integrating with respect to one variable the other variables are taken as constants and the Constant of Integration will be the function of variable which is treated as constant.

Eg: If $z = z(x, y)$, while Integrating w.r.t x. y is treated as constant and constant of Integration will a function $y \to f(y)$

Eg: ① $\frac{\partial z}{\partial x} = \sin x \cos y + xy^2 + x^2$

z is obtained by Integrating w.r to x treating y as constant.

$$z = -\cos y \cos x + y^2 \frac{x^2}{2} + \frac{x^3}{3} + f(y)$$

Eg: ② $\frac{\partial^2 z}{\partial x \partial y} = \sin x \cos y + xy^2 + x^2$

integr w.r.t x

$$z = -\cos y \cos x + y^2 \frac{x^2}{2} + \frac{x^3}{3} + f(y)$$

$$z = -\sin y \cos x + \frac{y^3}{3} \cdot \frac{x^2}{2} + \frac{x^3}{3} + f(y)$$

$$z = -\sin y \cos x + \frac{xy^3}{6} + x^3 y + g(x) + \int f(y) \cdot dy \; //$$

1. $\frac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x-y) = 0$

$$\frac{\partial^3 z}{\partial x^2 \partial y} = -18xy^2 - \sin(2x-y)$$

Integrating wrt 'x'. (treating y as const)

$$\frac{\partial^2 z}{\partial x \partial y} = -18y^2 \cdot \frac{x^2}{2} + \frac{\cos(2x-y)}{2} + f(y)$$ — integ const

Integrating wrt 'x'

$$\frac{\partial z}{\partial y} = -9y^2 \frac{x^3}{3} + \frac{\sin(2x-y)}{4} + x f(y) + g(y)$$

Integrating wrt y.

$$z = -3x^3 \frac{y^3}{3} - \frac{\cos(2x-y)}{-4} + x\int f(y)\,dy + \int g(y)\,dy + h(x)$$

2. $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} + a$

Integ wrt x

$$\frac{\partial z}{\partial y} = \frac{1}{y} \times \frac{x^2}{2} + ax + f(y)$$

Integ wrt y

$$z = \frac{x^2}{2} \log y + axy + \int f(y)\,dx + g(x)$$

3. $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos(2x+3y)$

Integrating wrt x

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\sin(2x+3y)}{2} + f(y)$$

Integrat wrt x

$$\frac{\partial z}{\partial y} = \frac{-\cos(2x+3y)}{4} + x f(y) + g(y)$$

Intg wrt y.

$$z = \frac{-\sin(2x+3y)}{12} + x\int f(y)\,dy + \int g(y)\,dy + h(x).$$

4. $\frac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$ given $u=0$ when $t=0$ and $\frac{\partial u}{\partial t} = 0$ at $x=0$ also Prove that $u \to \sin x$ $t \to \infty$

$$\frac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$$

Intg wrt 'x'

$$\frac{\partial u}{\partial t} = e^{-t}\sin x + f(t)$$

Given, $\frac{\partial u}{\partial t} = 0$ at $x=0$

$$0 = e^{-t}\cdot 0 + f(t) \quad \therefore f(t) = 0$$

$$\frac{\partial u}{\partial t} = e^{-t}\sin x$$

$\frac{\partial u}{\partial t} = e^{-t} \sin x$.

Integ wrt t

$u = -e^{-t} \sin x + g(x)$

Given $u = 0$ when $t = 0$

$0 = -1 \sin x + g(x) \Rightarrow \therefore g(x) = \sin x$.

$\therefore$ Soln : $u = -e^{-t} \sin x + \sin x$

$u = \sin x [1 - e^{-t}]$

As $t \to \infty$, $e^{-t} \to 0$

$\therefore u = \sin x (1 - e^{-t}) \to \sin x$.

* 5. $\frac{\partial^2 z}{\partial x^2} = xy$ $\quad \frac{\partial z}{\partial x} = \log(1+y)$ $x = 1$

$z = 0$ when $x = 0$.

$\frac{\partial z}{\partial x} = \frac{x^2 y}{2} + f(y)$ ← Sub

Given : $\frac{\partial z}{\partial x} = \log(1+y)$ when $x = 1$

$\therefore \log(1+y) = \frac{y}{2} + f(y)$ $\quad f(y) = \log(1+y) - \frac{y}{2}$

$\therefore \frac{\partial z}{\partial x} = \frac{x^2 y}{2} + \log(1+y) - \frac{y}{2}$

$z = \frac{x^3 y}{6} + x \log(1+y) - \frac{xy}{2} + g(y)$

Given $z=0$ when $x=0$

$$0 = 0+0-0+g(y) \qquad \therefore g(y) = 0.$$

$$\therefore z = \frac{x^2y}{6} + x\log(1+y) - \frac{xy}{2}.$$

6. $\dfrac{\partial^2 z}{\partial x\,\partial y} = \dfrac{x}{y}$ $\qquad \dfrac{\partial z}{\partial x} = \log x$ when $y=1$

$z=0$ when $x=1$

Integ wrt y

$$\frac{\partial z}{\partial x} = x\log y + f(x) \quad (\log 1 = 0)$$

given $\frac{\partial z}{\partial x} = \log x$ ← sub

$$\log x = f(x)$$

Integ wrt x

~~$z = \frac{x^2}{2}\log y + x f(x) + g(x)$.~~

$$z = \frac{x^2}{2}\log y + x f(x) - x + g(y).$$

$$z = \frac{x^2}{2}\log y + x\log x - x + g(y)$$

given when $x=1$, $z=0$

$$0 = \frac{\log y}{2} + 0 - 1 + g(y)$$

$$g(y) = 1 - \frac{\log y}{2}$$

$$z = \frac{x^2}{2}\log y + x\log x - x + 1 - \frac{\log y}{2}$$

Solution of homogeneous PDE with function of only one independent variable.

Given a homogeneous PDE with only one independent variable we assume that the dependent variable is function of only one variable and solve the resulting Ordinary diff Eqn (ODE) by the known method then replace the arbitrary constants by the arbitrary function of the variable ie not present in the given equation.

1. $\frac{\partial^2 z}{\partial x^2} + z = 0$ at $x=0$, $z = e^y$

$\frac{\partial z}{\partial x} = 1$

Let z be fn of only x

Then $\frac{\partial^2 z}{\partial x^2} \cdot \frac{d^2 z}{dx^2}$

$\frac{d^2 z}{dx^2} + z = 0$

$(D^2+1) z = 0$

AE : $D^2+1 = 0$

$D^2 = -1$

$D = \pm i$

$\therefore z = c_1 \cos x + c_2 \sin x$

but $z = z(x,y)$

dire Intg
funct of independent variable

soln

$$z = c_1(y)\cos x + c_2(y)\sin x \quad \text{--- (1)}$$

$$\frac{\partial z}{\partial x} = -c_1(y)\sin x + c_2(y)\cos x \quad \text{--- (2)}$$

Given $z = e^y$ at $x=0$ $\xrightarrow{(1)}$ $e^y = c_1(y)\cdot 1 + 0$

sub.

$\frac{\partial z}{\partial x} = 1$ at $x=0$ $\xrightarrow{(2)}$ $1 = 0 + c_2(y)$

soln $z = e^y\cos x + \sin x$

2. $\frac{\partial^2 z}{\partial y^2} = z$ when $y=0$, $z = e^x$

$y=0$, $\frac{\partial z}{\partial y} = e^{-x}$

$$\frac{\partial^2 z}{\partial y^2} - z = 0$$

let z be fn of only y

then DE becomes

$$\frac{d^2 z}{dy^2} - z = 0$$

$$(D^2-1)z = 0$$

AE = $D^2 - 1 = 0$

$D^2 = 1$

$D = \pm 1$

$$z = c_1 e^y + c_2 e^{-y}$$

But Z is f^n of both x & y

$\therefore Z = c_1(x)e^y + c_2(x)e^{-y}$

$\frac{\partial Z}{\partial y} = c_1(x)e^y - c_2(x)e^{-y}$

Given :- $Z = e^x$ at $y=0$

$e^x = c_1(x) + c_2(x)$.

Given :- $\frac{\partial Z}{\partial y} = e^{-x}$ at $y=0$.

$e^{-x} = c_1(x) - c_2(x)$.

$c_1(x) + c_2(x) = e^x \rightarrow (1)$

$c_1(x) - c_2(x) = e^{-x} \rightarrow (2)$

Add (1) & (2)

$c_1(x) + c_2(x) + c_1(x) - c_2(x) = e^x + e^{-x}$

$2c_1(x) = e^x + e^{-x}$

$c_1(x) = \frac{e^x + e^{-x}}{2} = \cosh x$

Sub (1) & (2)

~~$c_1(x) + c_2(x) - c_1(x) - c_2(x) = e^x - e^{-x}$~~

$c_1(x) + c_2(x) - c_1(x) + c_2(x) = e^x - e^{-x}$

$c_2(x) = \frac{e^x - e^{-x}}{2} = \sinh x$

$\therefore Z = \cosh x\, e^y + \sinh x\, e^{-y}$

Applications of Partial Differential Equations

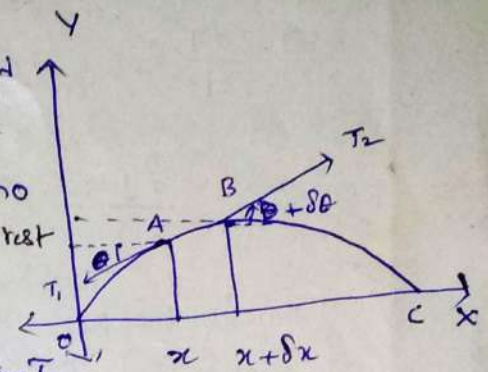
Derivation of 1-D wave Equation

consider a flexible string tightly stretched between two fixed points at a distance l apart. Let ρ be the mass per unit length of the string. We study the motion of the string with no External forces acting on it. Let the string be released from rest and allowed to vibrate.

We assume the following,

1. The tension T of the string is same throughout.
2. The effect of gravity can be ignored due to large tension T
3. The motion of the string is in small transverse vibrations [i.e, each point on the string makes small vibrations at right angles to the equilibrium position o x of the string Entirely in XY plane]

* [The tension T will be considered to be large as compared to the weight of the string so that the effects of gravity is negligible]



Taking o as origin, ox as x-axis, oy-as y-axis the motion takes place in Entirely in xy-plane. The figure shows the string in the position oABC at some time t.

consider an Element $A(x,y)$ $B(x+\delta x, y+\delta y)$ of the motion of the string where the tangents at A, B makes an angles θ, $\theta+\delta\theta$ with x-axis. Let $AB=\delta x$. and T_1, T_2 be the Tensions in the string at points A, B. $y(x,t)$ be the transverse displacement due to vibrations at a point x at time $t>0$.

Since there is no tension in the horizontal direction, the horizontal components T_1, T_2 must cancel Each other.

$$\therefore T_1 \cos\theta = T_2 \cos(\theta+\delta\theta) = T.$$

The vertical components of the Tensions at A is $-T_1 \sin\theta$ and at B is $T_2 \sin(\theta+\delta\theta)$

The -ve Sign is due to T_1 is directed downwards.

Hence the resultant tension acting vertically upwards $= T_2 \sin(\theta+\delta\theta) - T_1 \sin\theta$.

$$= T\frac{\sin(\theta+\delta\theta)}{\cos(\theta+\delta\theta)} - T\frac{\sin\theta}{\cos\theta} = T[\tan(\theta+\delta\theta) - \tan\theta]$$

$$= T\left[\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

Newton's second Law of motion, Force = mass x acc^n

$$\Rightarrow \rho.\delta x.\frac{\partial^2 y}{\partial t^2} = T\left[\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m} \frac{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\delta x}$$

Taking the limits as $B \to A$ i.e $\delta x \to 0$ we get

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = \frac{T}{m} \frac{\partial^2 y}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{where } c^2 = \frac{T}{m}$$

This is the PDE giving the transverse vibrations of the string. It is also known as wave Equation.

Various possible solutions of the one dimensional wave Equation $u_{tt} = c^2 u_{xx}$ by the method of Separation of Variables.

consider the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Let $u = X(x)\, T(t)$ be the soln

∴ we have $X \frac{d^2T}{dt^2} = c^2 T \frac{d^2X}{dx^2}$

$\div c^2 XT$ $\quad \frac{1}{c^2 T} \frac{d^2T}{dt^2} = \frac{1}{X} \frac{d^2X}{dx^2} \Rightarrow \frac{1}{c^2} \frac{T''}{T} = \frac{X''}{X} = K$ say

$\frac{X''}{X} = K \Rightarrow X'' - KX = 0 \quad \to (1)$

$\frac{1}{c^2}\frac{T''}{T} = K \Rightarrow T'' - c^2 K T = 0 \quad \to (2)$

Solving the equations (1) & (2) we get

1. when $K = p^2$ (say) $\quad X = C_1 e^{px} + C_2 e^{-px}$: $T = C_3 e^{cpt} + C_4 e^{-cpt}$
2. $K = -p^2$ $\quad X = C_5 \cos px + C_6 \sin px$: $T = C_7 \cos cpt + C_8 \sin cpt$
3. $K = 0$ $\quad X = C_9 x + C_{10}$: $T = C_{11} t + C_{12}$

∴ The various possible solutions of wave equation are,

$u = (C_1 e^{px} + C_2 e^{-px})(C_3 e^{cpt} + C_4 e^{-cpt})$

$u = (C_5 \cos px + C_6 \sin px)(C_7 \cos cpt + C_8 \sin cpt)$

$u = (C_9 x + C_{10})(C_{11} t + C_{12})$

Of these three solutions the second solution is most suited with the physical nature of the problem. As we are dealing with the problems on vibrations, u must be a periodic function of x and t. Hence the soln involving trig. functions is most suited.

∴ Soln of wave Equation $u(x,t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt)$

D'Alembert's Solution of the 1-D Wave Equation

Consider 1-D wave Equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \rightarrow (1)$

Let $v = x + ct$: $w = x - ct$

Now treating u as a function of v, w then v, w are fn of x & y.

We have $u = u(v, w)$:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\right] = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial v}\right] + \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial w}\right]$$

$$= \frac{\partial}{\partial v}\left(\frac{\partial u}{\partial v}\right)\cdot\frac{\partial v}{\partial x} + \frac{\partial}{\partial w}\left(\frac{\partial u}{\partial w}\right)\cdot\frac{\partial v}{\partial x} + \frac{\partial}{\partial w}\cdot\left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial w}\right)\frac{\partial w}{\partial x} \rightarrow (2)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial v^2} + 2\frac{\partial^2 u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w^2} = u_{vv} + 2u_{vw} + u_{ww}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial v}\cdot\frac{\partial v}{\partial t} + \frac{\partial u}{\partial w}\cdot\frac{\partial w}{\partial t}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial v}\cdot c - c\frac{\partial u}{\partial w}$$

$$\frac{\partial u}{\partial t} = c\left[\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right]$$

$$\frac{\partial^2 u}{\partial t^2} = c\,\frac{\partial}{\partial t}\left[\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right] = c\left[\frac{\partial}{\partial v}\left[\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right]\cdot\frac{\partial v}{\partial t} + \frac{\partial}{\partial w}\left[\frac{\partial u}{\partial v} - \frac{\partial u}{\partial w}\right]\cdot\frac{\partial w}{\partial t}\right]$$

$$= c^2\left[\frac{\partial^2 u}{\partial v^2} - \frac{\partial^2 u}{\partial v \partial w} - \frac{\partial^2 u}{\partial v \partial w} + \frac{\partial^2 u}{\partial w^2}\right] = c^2\left[u_{vv} - 2u_{vw} + u_{ww}\right] \rightarrow (3)$$

∴ Sub. (2) & (3) in (1) $c^2[u_{vv} - 2u_{vw} + u_{ww}] = c^2[u_{vv} + 2u_{vw} + u_{ww}]$

$\Rightarrow 4u_{vw} = 0 \Rightarrow u_{vw} = 0 \Rightarrow \frac{\partial}{\partial w}\left(\frac{\partial u}{\partial v}\right) = 0$

$\Rightarrow \frac{\partial u}{\partial v} = f(v) \Rightarrow u = \int f(v)\,dv + G(w) = F(v) + G(w)$

$u = F(x+ct) + G(x-ct)$ is the G.S. of wave Equation.

Now to determine F & G. Let initially $u(x,0) = f(x)$; $\left(\frac{\partial u}{\partial t}\right)_{(x,0)} = 0$

$$\frac{\partial u}{\partial t} = cF' - cG' \Rightarrow \left(\frac{\partial u}{\partial t}\right)_{(x,0)} = 0$$

$$\Rightarrow cF'(x+0) - cG'(x-0) = 0 \Rightarrow F'(x) = G'(x)$$

$$\Rightarrow F(x) = G(x) + K$$

$$u(x,0) = f(x) \Rightarrow f(x) = F(x) + G(x)$$

$$= G(x) + K + G(x)$$

$$\Rightarrow f(x) = 2G(x) + K \Rightarrow G(x) = \frac{1}{2}[f(x) - K]$$

$$\therefore F(x) = \frac{1}{2}(f(x) + K) + K = \frac{1}{2}(f(x) + K).$$

$$\therefore u = \frac{1}{2}[f(x+ct) + K + f(x-ct) - K]$$

$$u = \frac{1}{2}[f(x+ct) + f(x-ct)]$$ This is the D'Alembert's soln of wave eqn

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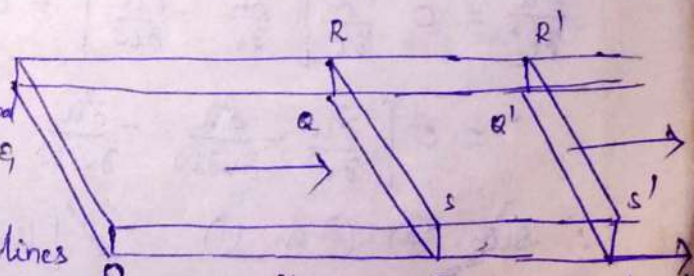
Derivation of 1D Heat Equation

We use the following fundamental principles involved in heat flow to derive 1-D Heat Equation.

1. Heat flows from a higher temperature to lower temperature.
2. The quantity of heat in a body is proportional to its mass & temp.
3. The rate of heat flow across an area is proportional to the area and to the temp. gradient normal to the area, where the constant of proportionality (K) is called the thermal conductivity.

Consider a homogeneous bar of uniform constant cross-sectional area A. Let ρ be the density, s be the specific heat & K be the thermal conductivity of the material.



Let the sides be insulated so that the stream lines of heat flow are parallel & perpendicular to A. Let one end of the bar be P taken as origin and direction of heat flow be the +ve X-axis.

Let $u = u(x,t)$ be the temperature of the slab at a distance x from origin.

Consider an element of bar between the planes PQRS and P'Q'R'S' at a distance x & x+δx from the end O. Let δu be the change in temparature in the slab of thickness δx of the bar.

The mass of the element = $A\rho\delta x$

The quantity of heat stored in this slab element = $As\rho\delta x\,\delta u$

Hence the rate of increase of heat in this slab element is $R = As\rho\delta x \cdot \frac{\partial u}{\partial t}$ → (1)

If R_i is the rate of inflow of heat & R_0 is the rate of outflow of heat then

$R_i = -KA\left(\frac{\partial u}{\partial x}\right)_x$: $R_0 = -KA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$ where -ve sign is due to (i)

→ (2)

∴ From (1) & (2) we have

$R = R_i - R_0$

$\Rightarrow As\rho\delta x \frac{\partial u}{\partial t} = KA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - KA\left(\frac{\partial u}{\partial x}\right)_x \Rightarrow \frac{\partial u}{\partial t} = \frac{K}{\rho s} \cdot \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x}$

taking limit as $\delta x \to 0$ we get

$\frac{K}{\rho s}\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \Rightarrow \frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$ This is the 1-D Heat Eq^n

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Various possible Solutions of 1-D Heat Equation $u_t = c^2 u_{xx}$ by the method of Separation of Variables.

Consider 1-D Heat Equation $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$

Let $u(x,t) = X(x)\,T(t)$ be the Soln

Then $X\frac{dT}{dt} = c^2 T\frac{d^2X}{dx^2} \Rightarrow \frac{1}{c^2T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = K$ (Let)

Then $\frac{1}{X}\frac{d^2X}{dx^2} = K$: $\frac{1}{c^2T}\frac{dT}{dt} = K$

$X'' - KX = 0$: $T' - Kc^2T = 0$.

Solving these Equations we get

case 1: when $K = p^2$ $X = c_1e^{px} + c_2e^{-px}$: $T = c_3e^{c^2p^2t}$

2: when $K = -p^2$ $X = c_4\cos px + c_5\sin px$: $T = c_6e^{-c^2p^2t}$

3: when $K = 0$ $X = c_7x + c_8$: $T = c_9$

Hence the various possible solutions of heat equation are,

$$u(x,t) = c_3 e^{c^2p^2t}\,(c_1 e^{px} + c_2 e^{-px})$$

$$u(x,t) = c_6 e^{-c^2p^2t}\,(c_4 \cos px + c_5 \sin px)$$

$$u(x,t) = c_9\,(c_7 x + c_8)$$

of these three solutions second solution is most suited for the physical nature of the problem. As we are dealing with the problem of heat conduction it must be a transient solution i.e, u is to decrease with increase of time 't'. Hence the solution of heat equation is, (most feasible)

$$u = (c_1 \cos px + c_2 \sin px)\, c_3\, e^{-c^2p^2t}$$

Module IV

Infinite Series and Power Series Solution

Series

Definitions

1) **Infinite Series**

If $u_1, u_2, u_3, \cdots, u_n, \cdots$ be an infinite sequence of real numbers, then

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots \infty$$

is called an **Infinite series**. An infinite series is denoted by $\sum_1 u_n$ and the sum of its first n terms is denoted by S_n

2) **Convergence, divergence and oscillation of a series**

Consider the infinite series $\sum_1 u_n = u_1 + u_2 + \cdots + u_n + \cdots \infty$ and let the sum of the first n terms be $S_n = u_1 + u_2 + \cdots + u_n$. Clearly S_n is a function of n and as n increases indefinitely three possibilities arise:

i) If S_n tends to a finite limit as $n \to \infty$, the series $\sum_1 u_n$ is said to be **convergent**.

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ii) If S_n tends to $\pm\infty$ as $n \to \infty$, the series $\sum u_n$ is said to be divergent

iii) If S_n does not tend to a unique limit as $n \to \infty$, then the series $\sum u_n$ is said to be oscillatory or non-convergent.

Examples:

1) Consider the series $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} + \cdots$

Ans Here $S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$

$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots \left(\frac{1}{n} - \frac{1}{n+1}\right)$.

$= 1 - \frac{1}{n+1}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$

Hence the series is Convergent

2) Consider the series $1^2 + 2^2 + 3^2 + \cdots$

Ans: Here $S_n = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6} = \infty$.

Here the series is divergent.

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iii) Consider the series $\sum (-1)^{n-1}$

Ans:

$S_n = 1 + (-1) + 1 + (-1) + \cdots$ upto n terms.

$= 1$ or 0 according as n is odd or even.

∴ The series oscillates finitely between 0 and 1.

II) D' Alembert's Ratio Test

In a positive term series $\sum u_n$ if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda$, then the series converges for $\lambda < 1$ and diverges for $\lambda > 1$

Note:

1) The Ratio test fails when $\lambda = 1$

eg: $\sum u_n = \sum \frac{1}{n^p}$.

Here $\lambda = \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left[\frac{1}{(n+1)^p} \cdot \frac{n^p}{1} \right]$

$= \lim_{n\to\infty} \left[\frac{1}{\left(1+\frac{1}{n}\right)^p} \right] = 1$

Then for all values of p, $\lambda = 1$; whereas $\sum \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p < 1$. Hence $\lambda = 1$ both for convergence and divergence of $\sum u_n$ which is absurd.

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Problems

1) Test for Convergence of the series $\sum \frac{(n^2-1)x^n}{(n^2+1)}$; $x>0$

Ans:

$$u_n = \frac{(n^2-1)x^n}{(n^2+1)}$$

$$u_{n+1} = \frac{[(n+1)^2-1]x^{n+1}}{(n+1)^2+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{[(n+1)^2-1]}{(n^2-1)} \cdot \frac{(n^2+1)}{[(n+1)^2+1]} \cdot \frac{x^{n+1}}{x^n}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left[\frac{(n+1)^2\left[1-\frac{1}{(n+1)^2}\right] n^2\left[1+\frac{1}{n^2}\right]}{n^2\left(1-\frac{1}{n^2}\right)(n+1)^2\left[1+\frac{1}{(n+1)^2}\right]} \right] x = x$$

→ When $x=1$, Test fails. $u_n = \frac{n^2-1}{n^2+1} = \frac{n^2\left(1-\frac{1}{n^2}\right)}{n^2\left(1+\frac{1}{n^2}\right)} \to 1$ as $n\to\infty$ $\therefore \sum u_n$ diverges

$\therefore$ By D'Alembert's Ratio test $\sum u_n$ converges if $x<1$ and diverges if $x>1$. and diverges of $x=1$

---

2) Test for Convergence the series:

$$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \cdots \infty$$

Ans: Here $u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$.

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$$u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{x^{2n-2}}{(n+1)\sqrt{n}} \times \frac{(n+2)\sqrt{n+1}}{x^{2n}}$$

$$= \lim_{n\to\infty} \left[\frac{(n+2)}{(n+1)}\left(\frac{n+1}{n}\right)^{1/2}\right] x^{-2}$$

$$= \lim_{n\to\infty} \left[\frac{n(1+\frac{2}{n})}{n(1+\frac{1}{n})} \frac{n^{1/2}(1+1/n)^{1/2}}{n^{1/2}}\right] x^{-2} = x^{-2}$$

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x^2$$

Hence $\sum u_n$ converges if $x^2 < 1$ i.e. $x^2 < 1$

and $\sum u_n$ diverges if $x^2 > 1$

If $x^2 = 1$ then $u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{3/2}(1+\frac{1}{n})}$

Taking $v_n = \frac{1}{n^{3/2}}$, we get $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$,

a finite qty $\therefore$, Both the $\sum u_n$ and $\sum v_n$ converge or diverge together. But $\sum v_n = \sum \frac{1}{n^{3/2}}$ is a convergent series, $\therefore$ $\sum u_n$ is also convergent.

Hence the given series converges if $x^2 \leq 1$ and diverges if $x^2 > 1$

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9) Discuss the convergence of the series

$$1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \cdots$$

Ans: Let $\sum u_n$ be the given series.

Then $u_n = \frac{n!}{n^n}$, $u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{n^n}{(n+1)^{n+1}} = \frac{(n+1)\cdot n^n}{(n+1)^{n+1}\left[1+\frac{1}{n}\right]^{n+1}}$$

$$= \lim_{n\to\infty} \frac{(n+1)}{n\left(1+\frac{1}{n}\right)^{n+1}} = \lim_{n\to\infty} \frac{n\left(1+\frac{1}{n}\right)}{n\left(1+\frac{1}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{(n+1)\,n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \cdot \frac{1}{\frac{(n+1)^n}{n^n}}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

∴ By D'Alembert's ratio test the series $\sum u_n$ is convergent.

———.

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4) Discuss the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n!}{(n^n)^2}$$

Ans: $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \frac{n^{2n}}{(n+1)^{2(n+1)}}$

$$= \lim_{n\to\infty} \frac{(n+1)\cdot n^{2n}}{(n+1)^{2n+2}} = \lim_{n\to\infty} \frac{n^{2n}}{(n+1)^{2n+1}}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{2n}(n+1)} = \frac{1}{e^2 \cdot \infty} = 0 < 1$$

Hence the given series is convergent.

5) Examine the convergence of the series:

i) $\frac{x}{1+x} + \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} + \cdots \infty$

Ans: $u_n = \frac{x^n}{1+x^n}$, $u_{n+1} = \frac{x^{n+1}}{1+x^{n+1}}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(1+x^{n+1})}{x^{n+1}} \cdot \frac{x^n}{(1+x^n)}$$

$$= \lim_{n\to\infty} \frac{(x^{n+1})}{(1+x^{n+1})} \times \frac{(1+x^n)}{x^n} = \lim_{n\to\infty} \frac{(x+x^{n+1})}{(1+x^{n+1})} = x$$

$(\because x^{n+1} \to 0 \text{ as } n\to\infty, \ x<1)$

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Also $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$ if $x = 1$

∴ By the ratio test $\sum_1^\infty u_n$ converges for $x < 1$

and fails for $x > 1$.

when $x = 1$, $\sum_1^\infty u_n = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$ which

is divergent.

Hence the given series converges for $x < 1$ and diverges for $x > 1$.

---

6) Examine the convergence of the series:

$$1 + \frac{(a+1)}{(b+1)} + \frac{(a+1)(2a+1)}{(b+1)(2b+1)} + \frac{(a+1)(2a+1)(3a+1)}{(b+1)(2b+1)(3b+1)} + \ldots$$

Ans: Neglecting the first term we have

$$u_{n+1} = u_n \cdot \frac{(na+1)}{(nb+1)}$$

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{u_n (na+1)}{(nb+1) u_n} = \lim_{n\to\infty} \frac{a + \frac{1}{n}}{b + \frac{1}{n}} = \frac{a}{b}$$

∴ By the ratio test $\sum_1^\infty u_n$ converges if $\frac{a}{b} < 1$

or $a < b$

and diverges for $a > b$ which is obvious

When $a = b$, the series looks $1 + 1 + 1 + 1 \ldots$ which

Hence the given series converges for $0 < a < b$ and diverges for $0 < b < a$.

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III) Cauchy's Root Test

In a positive series $\sum u_n$, if $\lim_{n\to\infty} (u_n)^{1/n} = \lambda$ then the series converges for $\lambda < 1$ and diverges for $\lambda > 1$.

Problems:

1) Discuss the convergence of the series whose n^{th} term is $\frac{n^{n^2}}{(n+1)^{n^2}}$ [Note: $(n^{n^2})^{1/n} = (n^{n\cdot n})^{1/n} = n^n$]

Ans:

$$(u_n)^{1/n} = \frac{n^n}{(n+1)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$\lim_{n\to\infty} u_n^{1/n} = \frac{1}{e} < 1$

$\therefore$ By Cauchy's root test the series is convergent ——.

2) Discuss the convergence of the series

i) $\sum \frac{n^3}{3^n}$ (ii) $\sum (\log n)^{-2n}$ (iii) $\sum \left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

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Ans:

i) $u_n = \frac{n^3}{3^n}$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left(\frac{n}{3}^{3/n}\right) = \lim_{n\to\infty} \frac{\left(n^{1/n}\right)^3}{3} = \frac{1}{3} < 1$$

$(\because \lim_{n\to\infty} n^{1/n} = 1)$.

Hence the given series converges by Cauchy's root test.

---

ii) $\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} (\log n)^{-2} = 0 \ (<1) \quad (\because \lim_{n\to\infty} \log n = 0]$

Hence by Cauchys root test the given series converges.

---

iii) $u_n = \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

$$(u_n)^{1/n} = \left[\frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{n^{3/2}}}\right]^{1/n} = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \frac{1}{e} < 1$$

Hence the given series is convergent

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3) Discuss the nature of the following series: (11)

i) $\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \cdots \infty \quad (x>0)$

ii) $\sum \frac{(n+1)^n x^n}{n^{n+1}}$

iii) $\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \cdots \infty$

Ans:

i) After leaving the first term we find that.

$u_n = \left(\frac{n+1}{n+2}\right)^n x^n$ so that

$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right) x = x.$

∴ By Cauchy's Root test, the given series converges for $x<1$ and diverges for $x>1$.

when $x=1$; $u_n = \left(\frac{n+1}{n+2}\right)^n = \frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}$

$\lim_{n\to\infty} (u_n)^{1/n} = \frac{1}{e} \neq 0$. Since u_n does not tend to zero, $\sum u_n$ is divergent

Thus the given series converges for $x<1$ and diverges for $x \geq 1$.

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ii) Here $(u_n)^{1/n} = \frac{(n+1)x}{n^{1+\frac{1}{n}}}$;

Note: The series $\sum \frac{1}{n^p}$ (Ex.
is convergent if $p > 1$
divergent if $p \leq 1$

$$\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \frac{(n+1)\,x}{n \cdot n^{1/n}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right) \cdot \frac{1}{n^{1/n}}\, x$$
$$= x$$

$\therefore$ The given series converges for $x < 1$
and diverges for $x \geq 1$.

when $x = 1$, $u_n = \frac{(n+1)^n}{n^{n+1}} = \frac{1}{n}\left(1+\frac{1}{n}\right)^n$

Taking $v_n = \frac{1}{n}$, $\lim_{n\to\infty} \left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \neq 0$ and finite

$\therefore$ By Comparison test both $\sum u_n$ and $\sum v_n$ behave alike. But $\sum v_n = \sum \frac{1}{n}$ is divergent ($\because p = 1$)
$\therefore \sum u_n$ is also divergent. Hence the given series converges for $x < 1$ and diverges for $x \geq 1$.

iii) Here $u_n = \left[\frac{(n+1)^{n+1}}{n^{n+1}} - \frac{n+1}{n}\right]^{-n}$.

$$\therefore (u_n)^{1/n} = \left(\frac{n+1}{n}\right)^{-1}\left[\left(\frac{n+1}{n}\right)^n - 1\right]^{-1} = \left(1+\frac{1}{n}\right)^{-1}\left[\left(1+\frac{1}{n}\right)^n - 1\right]^{-1}$$

$$\lim_{n\to\infty} u_n^{1/n} = 1 \cdot (e-1)^{-1} = \frac{1}{e-1} < 1$$

$\therefore$ The given series converges.

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Bessel's Equation

One of the most important differential equations in applied mathematics is

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \longrightarrow (1)$$

which is known as Bessel's equation of order n. Its particular solutions are called Bessel functions or cylindrical functions of order n. Many physical problems involving vibrations or heat conduction in cylindrical regions give rise to this equation.

Substituting

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots, \quad a_0 \neq 0 \longrightarrow (A)$$

eq (1) takes the form

$$a_0(m^2 - n^2)x^m + a_1\{(m+1)^2 - n^2\}x^{m+1} + \{a_2[(m+2)^2 - n^2] + a_0\}x^{m+2} + \{a_3[(m+3)^2 - n^2] + a_1\}x^{m+3} + \cdots = 0$$

$$\left[\because y_1 = a_0 m x^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + \cdots\right.$$

$$\left. y_2 = a_0 m(m-1)x^{m-2} + a_1(m+1)m x^{m-1} + a_2(m+2)(m+1)x^m + \cdots\right]$$

$$\left[\because x^2\left[a_0(m^2 - m)x^{m-2} + a_1(m+1)m x^{m-1} + a_2(m+2)(m+1)x^m + \cdots\right]\right.$$

$$\left. + x\left[a_0 m x^{m-1} + a_1(m+1)x^m + a_2(m+2)x^{m+1} + \cdots\right] + (x^2 - n^2)\left(a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots\right) = 0\right]$$

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Equating to zero the co-efficient of x^m, we obtain the indical equation $m^2 - n^2 = 0$ (as $a_0 \neq 0$)

where $m = n$ or $-n$. Equating the coeff. of x^{m+1} to zero we get $a_1[(m+1)^2 - n^2] = 0$, but $(m+1)^2 - n^2 \neq 0 \Rightarrow a_1 = 0$

$\Rightarrow a_1 = a_3 = a_5 = a_7 = \cdots = 0$

and $$a_2 = -\frac{a_0}{(m+2)^2 - n^2}$$

$$a_4 = -\frac{a_2}{(m+4)^2 - n^2} \quad \text{etc.}$$

These give from eq (A)

$$y = a_0 x^m \left[1 - \frac{1}{(m+2)^2 - n^2} x^2 + \frac{1}{[(m+2)^2 - n^2][(m+4)^2 - n^2]} x^4 - + \cdots\right]$$

For $m = n$, we get

$$y_1 = a_0 x^n \left\{1 - \frac{1}{4(n+1)} x^2 + \frac{1}{4^2 2!\,(n+1)(n+2)} x^4 - \frac{1}{4^3 3!\,(n+1)(n+2)(n+3)} x^6 + \cdots\right\} \rightarrow (2)$$

and for $m = -n$, we have

$$y_2 = a_0 x^{-n} \left\{1 - \frac{1}{4(-n+1)} x^2 + \frac{1}{4^2 \cdot 2!\,(-n+1)(-n+2)} x^4 - \frac{1}{4^3\, 3!\,(-n+1)(-n+2)(-n+3)} x^6 + \cdots\right\}$$

$\hookrightarrow (3)$

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Case 1: When n is not integral or zero, the complete solution of (1) is

$$y = c_1 y_1 + c_2 y_2$$

If we take $a_0 = \frac{1}{2^n \Gamma(n+1)}$, then the solution given by (2) is called the Bessel function of the first kind of order n and is denoted by $J_n(x)$. Thus

$$J_n(x) = \left(\frac{x}{2}\right)^n \left\{ \frac{1}{\Gamma(n+1)} - \frac{1}{1!\,\Gamma(n+2)} \left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma(n+3)} \left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\Gamma(n+4)} \left(\frac{x}{2}\right)^6 + \cdots \right\}$$

$(n>0)$

i.e. $$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,\Gamma(n+r+1)} \rightarrow (4)$$

and corresponding to equation (3) we have

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \frac{1}{r!\,\Gamma(-n+r+1)} \rightarrow (5)$$

which is called the Bessel function of the first kind of order $-n$.

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Hence complete solution of the Bessel equation may be expressed in the form.

$$y = A J_n(x) + B J_{-n}(x) \longrightarrow (6)$$

Orthogonality of Bessel Functions

We shall prove that

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = \begin{cases} 0, & \alpha \neq \beta \\ \frac{1}{2}[J_{n+1}(\alpha)]^2, & \alpha = \beta \end{cases}$$

where α and β are roots of $J_n(x) = 0$.

We know that the solution of the equation

$$x^2 u'' + x u' + (\alpha^2 x^2 - n^2) u = 0 \longrightarrow (1)$$

$$x^2 v'' + x v' + (\beta^2 x^2 - n^2) v = 0 \longrightarrow (2)$$

and $u = J_n(\alpha x)$, $v = J_n(\beta x)$.

Multiplying eq (1) by $\frac{v}{x}$ and equation (2) by $\frac{u}{x}$ and subtracting we get

$$x(u''v - uv'') + (u'v - uv') + (\alpha^2 - \beta^2) x u v = 0$$

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or

$$\frac{d}{dx}\left[x(u'v - uv')\right] = (\beta^2 - \alpha^2)\, x\, uv$$

Now integrating both sides from 0 to 1.

$$(\beta^2 - \alpha^2)\int_0^1 x\, uv\, dx = \left[x(u'v - uv')\Big|_0^1\right] \quad \text{--- (3)}$$

$$= (u'v - uv')_{x=1}$$

Since $u = J_n(\alpha x)$

$$u' = \frac{d}{dx}\left[J_n(\alpha x)\right] = \frac{d\left[J_n(\alpha x)\right]}{d(\alpha x)} \cdot \frac{d(\alpha x)}{dx}$$

$$= \alpha\, J_n'(\alpha x)$$

Similarly, $v = J_n(\beta x)$ and $v' = \beta\, J_n'(\beta x)$.

Substituting these in (3) we get

$$\int_0^1 x\, J_n(\alpha x)\, J_n(\beta x)\, dx = \frac{\left[\alpha J_n'(\alpha) J_n(\beta) - \beta J_n(\alpha) J_n'(\beta)\right]}{\beta^2 - \alpha^2} \quad \text{--- (4)}$$

If α and β are distinct roots of $J_n(x) = 0$ then $J_n(\alpha) = J_n(\beta) = 0$ and equation (4) reduces to

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$$\int_0^1 x\, J_n(\alpha x)\, J_n(\beta x)\, dx = 0 \qquad \rightarrow (5)$$

This is known as the orthogonality relation of Bessel's function.

When $\beta = \alpha$, the R.H.S of eq (4) is of the form $\frac{0}{0}$. Its value can be found by considering α as a root of $J_n(x) = 0$ $\Rightarrow J_n(\alpha) = 0$ and β as a variable approaching α. Then eq (4) gives.

$$\lim_{\beta \to \alpha} \int_0^1 x\, J_n(\alpha x)\, J_n(\beta x)\, dx$$

$$= \lim_{\beta \to \alpha} \frac{\alpha\, J_n'(\alpha)\, J_n(\beta)}{\beta^2 - \alpha^2}$$

or by L' Hospital's rule,

$$\int_0^1 x\, J_n^2(\alpha x)\, dx = \lim_{\beta \to \alpha} \frac{\alpha\, J_n'(\alpha)\, J_n'(\beta)}{2\beta}$$

$$= \frac{1}{2}\left[J_n'(\alpha)\right]^2 = \frac{1}{2}\left[J_{n+1}(\alpha)\right]^2 \rightarrow (6)$$

$$\left[\because J_n'(\alpha) = \frac{n}{\alpha} J_n(\alpha) - J_{n+1}(\alpha)\right]$$

$$\left[\because \alpha \text{ is a root of } J_n(x) = 0 \Rightarrow J_n(\alpha) = 0\right]$$

Note: $\int_0^1 x\, J_n^2(\alpha x)\, dx = \frac{1}{2}\left[J_n'(\alpha)\right]^2$

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Problems

1) Prove that $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$.

Ans:

We have

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} \cdot \frac{1}{r!\,\Gamma(-n+r+1)}$$

$$\therefore J_{-1/2}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{\left(\frac{x}{2}\right)^{-1/2+2r}}{r!\,\Gamma(-1/2+r+1)}$$

$$\text{ie } J_{-1/2}(x) = \frac{\left(\frac{x}{2}\right)^{-1/2}}{\Gamma(1/2)} - \frac{\left(\frac{x}{2}\right)^{3/2}}{1!\,\Gamma(3/2)} + \frac{\left(\frac{x}{2}\right)^{7/2}}{2!\,\Gamma(5/2)} - + \cdots$$

$$= \left(\frac{x}{2}\right)^{-1/2}\left[\frac{1}{\Gamma(1/2)} - \frac{\left(\frac{x}{2}\right)^{2}}{1\cdot\frac{1}{2}\Gamma(1/2)} + \frac{\left(\frac{x}{2}\right)^{4}}{1\cdot 2\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)} - + \cdots\right]$$

$(\because \Gamma(n+1) = n\Gamma(n))$

$$= \frac{\left(\frac{x}{2}\right)^{-1/2}}{\Gamma(1/2)}\left[1 - \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - + \cdots\right]$$

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$$= \frac{\left(\frac{x}{2}\right)^{-1/2}}{\Gamma(1/2)} \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots\right]$$

$$= \frac{\sqrt{2}}{\sqrt{\pi x}} \cos x$$

2) Prove that- $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$.

Ans:

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \cdot \frac{1}{r!\,\Gamma(n+r+1)}$$

$$J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{1/2+2r}}{r!\,\Gamma(1/2+r+1)}$$

$$= \frac{\left(\frac{x}{2}\right)^{1/2}}{\Gamma(3/2)} - \frac{\left(\frac{x}{2}\right)^{5/2}}{1!\,\Gamma(5/2)} + \frac{\left(\frac{x}{2}\right)^{9/2}}{2!\,\Gamma(7/2)} - + \cdots$$

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$$= \left(\frac{x}{2}\right)^{1/2}\left[\frac{1}{\frac{1}{2}\Gamma(1/2)} - \frac{\left(\frac{x}{2}\right)^2}{1!\,\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)} + \frac{\left(\frac{x}{2}\right)^4}{2!\,\frac{5}{2}\cdot\frac{3}{2^2}\Gamma(1/2)} - + \cdots\right]$$

$$= \left(\frac{x}{2}\right)^{1/2}\left[\frac{1}{\frac{1}{2}\Gamma(1/2)}\right]$$ Take $\frac{1}{2}\Gamma(1/2)$ out common & × by x.

$$= \frac{\sqrt{x}}{\sqrt{2}}\cdot\frac{2}{\Gamma(1/2)}\cdot\frac{1}{\sqrt{x}}\left[x - \frac{x^3}{2^2\cdot\frac{3}{2^2}} + \frac{x^5}{\frac{1\cdot3\cdot5}{2^2}\,2^5} - + \cdots\right]$$

$$= \frac{\sqrt{2}}{\sqrt{\pi x}}\cdot\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots\right]$$

$$= \frac{\sqrt{2}}{\sqrt{\pi x}}\sin x$$

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Legendre's Differential Equation

Another differential equation of importance in Applied Mathematics, particularly in boundary value problems for spheres is Legendre's equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \rightarrow (1)$$

Here n is a real number. But in most applications only integral values of n are required.

Substituting,

$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \cdots \quad (a_0 \neq 0).$$

Eq (1) takes the form.

$$a_0m(m-1)x^{m-2} + a_1(m+1)mx^{m-1} + \cdots + \left[a_{r+2}(m+r+2)(m+r+1) - \{(m+r)(m+r+1) - n(n+1)\}a_r\right]x^{m+r} + \cdots = 0$$

Equating to zero the coefficient of the lowest power of x i.e. of x^{m-2}, we get.

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$a_0 m(m-1) = 0$, $m = 0, 1$ $(\because a_0 \neq 0,$

Equating to zero the coefficients of x^{m-1} and x^{m+r}, we get $a_1 (m+1) m = 0 \rightarrow (2)$

$$a_{r+2}(m+r+2)(m+r+1) - \{(m+r)(m+r+1) - n(n+1)\} a_r = 0$$

when $m=0$, eq (2) is satisfied $\rightarrow (3)$

and therefore $a_1 \neq 0$. Then (3) gives, taking $r = 0, 1, 2, \ldots$ in turn.

$$a_2 = -\frac{n(n+1)}{2!} a_0 , \quad a_3 = -\frac{(n-1)(n+2)}{3!} a_1$$

$$a_4 = -\frac{(n-2)(n+3)}{4 \cdot 3} a_2 = \frac{n(n-2)(n+1)(n+3)}{4!} a_0$$

$$a_5 = -\frac{(n-3)(n+4)}{5 \cdot 4} a_3 = \frac{(n-1)(n-3)(n+2)(n+4)}{5!} a_1$$

Hence for $m=0$, there are two independent solutions of eq (1), etc.

$$y_1 = a_0 \left\{1 - \frac{n(n+1)}{2!} x^2 + \frac{(n-2)n(n+1)(n+3)}{4!} x^4 + \cdots\right\} \rightarrow (4)$$

$$y_2 = a_1 \left\{x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - + \cdots\right\} \rightarrow (5)$$

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when $m=1$, equation (3) shows that $a_1 = 0$.

Therefore, equation (4) gives

$$a_3 = a_5 = a_7 = \cdots = 0 \quad \text{and}$$

$$a_2 = -\frac{(n-1)(n+2)}{3!}\, a_0$$

$$a_4 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!}\, a_0 \quad \text{etc.}$$

Thus for $m=1$, we get the solution

$$y_2 = a_0 x \left[1 - \frac{(n-1)(n+2)}{3!}x^2 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^4 - \cdots\right] \quad \text{(5)}$$

again.

Hence $y = y_1 + y_2$ is the general solution of (1).

If n is a positive even integer, the series (4) terminates at the term in x^n and y_1 becomes a polynomial. Similarly if n is an odd integer, (5) becomes a polynomial of degree n. Thus whenever n is a positive integer the general solution of eq (1) consists of a polynomial solution and an infinite series solution. Those polynomial solutions with a_0 or a_1 so chosen that the value of the polynomial is 1 for $x=1$ are called Legendre polynomials of order n and are denoted by $P_n(x)$. The infinite series solution with (a_0 or a_1 properly chosen) is called Legendre function of the second kind and is denoted by $Q_n(x)$.

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Rodrigue's Formula

$$P_n(x) = \frac{1}{n!\,2^n} \frac{d^n}{dx^n} (x^2-1)^n \quad \rightarrow (1).$$

Legendre Polynomials

Using eq (1) we get.

$P_0(x) = 1$; $P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$, $P_3(x) = \frac{1}{2}(5x^3-3x)$

$P_4(x) = \frac{1}{8}[35x^4 - 30x^2 + 3]$.

$P_5(x) = \frac{1}{8}[63x^5 - 70x^3 + 15x]$.

→ when $n=0$ in (1)

$$P_0(x) = \frac{1}{0!\,2^0} \frac{d^0}{dx^0} (x^2-1)^0 = 1$$

→ when $n=1$ in (1),

$$P_1(x) = \frac{1}{1!\,2^1} \frac{d}{dx} (x^2-1)^1 = \frac{1}{2}(2x) = x$$

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→ when $n = 2$ in (1).

$$P_2(x) = \frac{1}{2!\,2^2} \frac{d^2}{dx^2}(x^2-1)^2$$

$$= \frac{1}{2(4)} \left(\frac{d}{dx} 2(x^2-1) \cdot 2x \right) = \frac{1}{8} \frac{d}{dx}(4x^3 - 4x)$$

$$= \frac{1}{8} \left[12x^2 - 4 \right] = \frac{1}{2}(3x^2 - 1)$$

→ when $n = 3$ in (1).

$$P_3(x) = \frac{1}{3!\,2^3} \frac{d^3}{dx^3}(x^2-1)^3$$

$$= \frac{1}{6 \times 8}(120x^3 - 72x) = \frac{1}{2}(5x^3 - 3x)$$

→ when $n = 4$ in (1).

$$P_4(x) = \frac{1}{4!\,2^4} \frac{d^4}{dx^4}(x^2-1)^4$$

$$= \frac{1}{24 \times 16}(1680x^4 - 1440x^2 + 144)$$

$$= \frac{1}{8}(35x^4 - 30x^2 + 3)$$

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→ when $n = 5$ in (1).

$$P_5(x) = \frac{1}{5!\,2^5} \frac{d^5}{dx^5}(x^2-1)^5$$

$$= \frac{1}{120 \times 32}(30240x^5 - 33600x^3 + 7200)$$

$$= \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

---

2) Express $x^3 - 5x^2 + 6x + 1$ in terms of Legendre Polynomials.

Ans: we have $P_3(x) = \frac{1}{2}(5x^3 - 3x) \Rightarrow 5x^3 = 2P_3(x) + 3x$

$$\Rightarrow x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}x = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \Rightarrow 3x^2 - 1 = 2P_2(x)$$

$$x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$$

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$\therefore \quad x^3 - 5x^2 + 6x + 1$

$= \left\{ \frac{2}{5} P_3(x) + \frac{3}{5} P_1(x) \right\} - 5 \left\{ \frac{2}{3} P_2(x) + \frac{1}{3} P_0(x) \right\}$

$+ 6 \{ P_1(x) \} + P_0(x)$

$= \frac{2}{5} P_3(x) - \frac{10}{3} P_2(x) + \frac{33}{5} P_1(x) - \frac{2}{3} P_0(x).$

$=$

3) Express $f(x) = x^3 + 2x^2 - x - 3$ in terms of Legendre polynomials.

Ans: $x^3 + 2x^2 - x - 3$

$= \left\{ \frac{2}{5} P_3(x) + \frac{3}{5} P_1(x) \right\} + 2 \left\{ \frac{2}{3} P_2(x) + \frac{1}{3} P_0(x) \right\}$

$- P_1(x) - 3 P_0(x).$

$= \frac{2}{5} P_3(x) + \frac{4}{3} P_2(x) - \frac{2}{5} P_1(x) - \frac{7}{3} P_0(x)$

$=$

4) Use Rodrigues formula to show that $P_3(\cos\theta) = \frac{1}{8}(3\cos\theta + 5\cos 3\theta)$

$=$

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Ans:

Using Rodrigues formula we have

$$P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

$$\therefore P_3(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta) \quad \hookrightarrow (1)$$

Note: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$$\Rightarrow \cos^3\theta = \frac{1}{4}[\cos 3\theta + 3\cos\theta] \rightarrow (2)$$

Substituting (2) in (1), we get.

$$P_3(\cos\theta) = \frac{1}{2}\left[\frac{5}{4}(\cos 3\theta + 3\cos\theta) - 3\cos\theta\right]$$

$$= \frac{1}{8}[5\cos 3\theta + 15\cos\theta - 12\cos\theta]$$

$$= \frac{1}{8}[3\cos\theta + 5\cos 3\theta]$$

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4) Use Rodrigues formula to show that.

$$P_4(\cos\theta) = \frac{1}{64}\left[35\cos 4\theta + 20\cos 2\theta + 9\right]$$

Ans: Using Rodrigues formula we have.

$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$

$$\therefore P_4(\cos\theta) = \frac{1}{8}\left(35\cos^4\theta - 30\cos^2\theta + 3\right)$$

$$= \frac{1}{8}\left(35\left(\cos^2\theta\right)^2 - 30\cos^2\theta + 3\right)$$

$$= \frac{1}{8}\left[35\left(\frac{1+\cos 2\theta}{2}\right)^2 - 30\left(\frac{1+\cos 2\theta}{2}\right) + 3\right]$$

$$= \frac{1}{8}\left[\frac{35}{4}\left(1 + 2\cos 2\theta + \cos^2 2\theta\right) - \frac{30}{2}\left(1+\cos 2\theta\right) + 3\right]$$

$$= \frac{1}{8}\left[\frac{35}{4} + \frac{35}{2}\cos 2\theta + \frac{35}{4}\left(\frac{1+\cos 4\theta}{2}\right) - \frac{30}{2}\left(1+\cos 2\theta\right) + 3\right]$$

$$= \frac{1}{8}\left[\frac{35}{4} + \frac{35}{8} - \frac{30}{2} + 3 + \frac{35}{2}\cos 2\theta + \frac{35}{8}\cos 4\theta - \frac{30}{2}\cos 2\theta\right]$$

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$$= \frac{1}{8}\left[\frac{35}{8}\cos 4\theta + \frac{5}{2}\cos 2\theta + \frac{9}{8}\right]$$

$$= \frac{1}{64}\left[35\cos 4\theta + 20\cos 2\theta + 9\right]$$

$=$

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Interpolation:

The word interpolation denotes the method of computing the value of the function $y = f(x)$ for any given value of x when a set $(x_0, y_0), (x_1, y_1), \ldots (x_n, y_n)$ are given.

Note:

Since in most of the cases the exact form of the function is not known. In such cases the function f(x) is replaced by a simpler function $\varphi(x)$ which has the same values as f(x) for $x_0, x_1, x_2 \ldots, x_n$.

$$\varphi(x) = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots.$$

$$+ \frac{u(u-1)(u-2)\ldots(u-n+1)}{n!}\Delta^n y_0$$

Note :

1. Newton forward interpolation is used to interpolate the values of y near the beginning of a set of tabular values.
2. y_0 may be taken as any point of the table but the formula contains those values of y which come after the value chosen as y_0.

Problems:

1) The table gives the distances in nautical miles of the visible horizon for the given heights in feet above the earths surface.

x = height	100	150	200	250	300	350	400
y = distance	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when i) x = 120, ii) x **=218**

Solution:

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
100	10.63						
		2.40					
150	13.03		-0.39				
		2.01		0.15			
200	15.04		-0.24		-0.07		
		1.77		0.08		0.02	
250	16.81		-0.16		-0.05		0.02
		1.61		0.03		0.04	
300	18.42		-0.13		-0.01		
		1.48		0.02			
350	19.90		-0.11				
		1.37					
400	21.27						

Choose $x_0 = 100$

i) $x = 120,\ u = \dfrac{120-100}{50} = 0.4$

$$f(120) = 10.63 + \frac{0.4}{1!}(2.40) + \frac{(0.4)(0.4-1)}{2!}(-0.39)$$

$$+ \frac{(0.4)\,(0.4-1)(0.4-2)}{3!}(0.15)$$

$$+ \frac{(0.4)(0.4-1)\,(0.4-2)\,(0.4-3)}{4!}(-0.07)$$

$$+ \frac{(0.4)\,(0.4-1)\,(0.4-2)(0.4-3)\,(0.4-4)}{5!}(0.02)$$

$$+ \frac{(0.4)\,(0.4-1)\,(0.4-2)\,(0.4-3)\,(0.4-4)\,(0.4-5)}{6!}(0.02) = 11.649$$

ii) Let $x = 218,\ x_0 = 200,\ u = \dfrac{218-200}{50} = \dfrac{18}{50} = 0.36$

$$f(218) = 15.04 + 0.36(1.77) + \frac{0.36\,(-0.64)}{2}(-0.16)$$

$$+ \frac{0.36(-0.64)\,(-1.64)}{6}(0.03) + \ldots$$

$$= 15.7$$

2) Find the value of f(1.85).

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.7	5.474						
		0.575					
1.8	6.049		0.062				
		0.637		0.004			
1.9	6.686		0.066		0.004		
		0.703		0.008		-0.004	
2.0	7.389		0.074		0		0.004
		0.777		0.008		0	
2.1	8.166		0.082		0		
		0.859		0.008			
2.2	9.025		0.090				
		0.949					
23	9.974						

$$\text{Choose } x_0 = 1.8,\ x = 1.85\ u = \frac{x - x_0}{h} = \frac{1.85 - 1.8}{0.1} = 0.5$$

$$f(1.85) = 6.049 + (0.5)(0.637) + \frac{(0.5)(-0.5)}{2}(0.066)$$

$$+ \frac{(0.5)(-0.5)(-1.5)}{6}(0.008)$$

$$= 6.049 + 0.3185 - 0.0008 + 0.0005$$

$$= 6.359$$

3) Given $\sin 45^o = 0.7071$, $\sin 50^o = 0.7660$, $\sin 55^o = 0.8192$, $\sin 60^o = 0.8660$. Find $\sin 48^o$.

x	y	Δ	Δ^2	Δ^3
45	0.7071			
		0.589		
50	0.7660		-0.0057	
		0.0532		0.0007
55	0.8192		-0.0064	
		0.0468		
60	0.8660			

$$x = 48,\ x_0 = 45;\ h = 5\ u = \frac{x - x_0}{h} = 0.6$$

$$\sin 48^o = 0.7071 + (0.6)\ (0.0589)$$

$$+\frac{(0.6)(-0.4)}{2}(-0.0057) + \frac{(0.6)(-0.4)\ (-1.4)}{6}(0.0007) = 0.7431$$

4) From the following data find the number of students who have obtained ≤ 45 marks. Also find the number of students who have scored between 41 and 45 marks.

Marks	**0 - 40**	**41 - 50**	**51 - 60**	**61 -70**	**71 - 80**
No. of students	**31**	**42**	**51**	**35**	**31**

x	y	Δ	Δ^2	Δ^3	Δ^4
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

$$f(45) = 31 + (0.5)(42) + \frac{(0.5)(-0.5)\,9}{2} + \frac{(0.5)(-0.5)(-1.5)(-25)}{3!}$$

$$+ \frac{(0.5)(-0.5)(-1.5)(-2.5)}{4!}(37) = 47.8672 \approx 48$$

(48-31=17)

f(45) - f(40) = 17 = Number of students who have scored between 41 and 45.

5) Find the interpolating polynomial for the following data:

f(0) = 1, f(1) = 0, f(2) = 1, f(3) = 10. Hence evaluate f(0.5)

x	y	Δ	Δ^2	Δ^3
0	1			
		-1		
1	0		2	
		1		6
2	1		8	
		9		
3	10			

$$u = \frac{x - 0}{1} = x$$

$$f(x) = 1 + x(-1) + \frac{x(x-1)}{2!}(2) + \frac{x(x-1)\,(x-2)}{3!}\,6 = x^3 - 2x^2 + 1$$

6) Find the interpolating polynomial for the following data:

x:	0	1	2	3	4
f(x) :	3	6	11	18	27

x	y	Δ	Δ^2	Δ^3	Δ^4
0	3				
		3			
1	6		2		
		5		0	
2	11		2		0
		7		0	
3	18		2		
		9			
4	27				

$$u = \frac{x-0}{1} = x$$

$$f(x) = 3 + x(3) + \frac{x(x-1)}{2}(2) + \frac{x(x-1)}{x!}(0) \qquad = 3 + 2x + x^2$$

Newton Gregory Backward Interpolation formula

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \ldots.$$

where $u = \frac{x - x_n}{h}$

1) **The values of tan x are given for values of x in the following table. Estimate tan (0.26)**

x	0.10	0.15	0.20	0.25	0.30
y	0.1003	0.1511	0.2027	0.2553	0.3093

x	y	∇	∇^2	∇^3	∇^4
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.0540			
0.30	0.3093				

$$u = \frac{0.26 - 0.3}{0.05} = -0.8$$

$$f(0.26) = 0.3093 + (-0.8)(0.054) + \frac{(-0.8)}{2}(0.2)(0.0014) + \frac{(-0.8)(0.2)(1.2)}{6}(0.0004) = 0.2659$$

2) **The deflection d measured at various distances x from one end of a cantilever is given by the following table. Find d when x = 0.95**

$u = \frac{0.95 - 1}{0.2} = -0.25$ d = 0.3308 when x = 0.95

x	d	∇	∇^2	∇^3	∇^4	∇^5
0	0					
		0.0347				
0.2	0.0347		0.0479			
		0.0826		-0.0318		
0.4	0.1173		0.0161		0.0003	
		0.0987		-0.0321		-0.0003
0.6	0.2160		-0.016		0	
		0.0827		-0.032		
0.8	0.2987		-0.0481			
		0.0346				
1.0	0.3333					

3) The area y of circles for different diameters x are given below:

x :	80	85	90	95	100
y :	5026	5674	6362	7088	7854

Calculate area when x = 98

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
80	5026				
		648			
85	5674		40		
		688		-2	
90	6362		38		4
		726		2	
95	7088		40		
		766			
100	7854				

Answer:

$$u = \frac{x - x_n}{h} = -0.4$$

$$y = 7542$$

4) Find the interpolating polynomial which approximates the following data.

x	0	1	2	3	4
y	-5	-10	-9	4	35

x	y	∇	∇^2	∇^3	∇^4
0	-5				
		-5			
1	-10		6		
		1		6	
2	-9		12		0
		13		6	
3	4		18		
		31			
4	35				

$$u = \frac{x-4}{1}$$

$$f(x) = 35 + (x-4)\,(31) + (x-4)\,(x-3)\frac{18}{2!} + \frac{(x-4)\,(x-3)\,(x-2)\,(6)}{3!}$$

$$\boxed{f(x) = x^3 + 2x^2 + 6x - 5}$$

Interpolation with unequal intervals

Newton backward and forward interpolation is applicable only when $x_0, x_1,\ldots,x_{n-1}$ are equally spaced. Now we use two interpolation formulae for unequally spaced values of x.

i) Lagranges formula for unequal intervals:

If $y = f(x)$ takes the values $y_0, y_1, y_2,\ldots,y_n$ corresponding to $x = x_0, x_1, x_2,\ldots,x_n$ then

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} f(x_0)$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)} f(x_1)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)} f(x_2)+\ldots.$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})} f(x_n)$$ is known as the lagrange's interpolation formula

Newton's divided difference interpolation formula

$$y = f(x) = y_0 + (x - x_0)\,\Delta y_0 + (x - x_0)\,(x - x_1)\,\Delta^2 y_0 + (x - x_0)\,(x - x_1)(x - x_2)\,\Delta^3 y_0$$

$$+ \ldots + (x - x_0)\,(x - x_1)\ldots(x - x_n)\,\Delta^n y_0$$

is called the Newton's divided difference formula.

Note: Lagrange's formula has the drawback that if another interpolation value were inserted, then the interpolation coefficients need to be recalculated.

Inverse interpolation: Finding the value of y given the value of x is called interpolation where as finding the value of x for a given y is called inverse interpolation.

Since Lagrange's formula is only a relation between x and y we can obtain the inverse interpolation formula just by interchanging x and y.

$$\therefore x = \frac{(y-y_1)(y-y_2)....(y-y_n)}{(y_0-y_1)(y_0-y_2)...(y_0=y_n)}.x_0$$

$$+\frac{(y-y_0)(y-y_0)(y-y_2)(y-y_3)...(y-y_n)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)...(y_1-y_n)}x_1+...$$

$$+...+\frac{(y-y_0)(y-y_1)...(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)...(y_n-y_{n-1})}.x_n$$

is the Lagranges formula for inverse interpolation

1) **The following table gives the values of x and y**

x :	**1.2**	**2.1**	**2.8**	**4.1**	**4.9**	**6.2**
y :	**4.2**	**6.8**	**9.8**	**13.4**	**15.5**	**19.6**

Find x when y = 12 using Lagranges inverse interpolation formula.

Using Langrages formula

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)(y-y_4)(y-y_5)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)(y_0-y_4)(y_0-y_5)} x_0$$

$$+ \ldots + \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_3)(y-y_4)}{(y_5-y_0)(y_5-y_1)(y_5-y_2)\ldots(y_5-y_4)} x_4$$

$$= 0.022 - 0.234 + 1.252 + 3.419 - 0.964 + 0.055$$

$$= 3.55$$

2) Given the values

x :	5	7	11	13	17
f(x) :	150	392	1452	2366	5202

Evaluate f(9) using (i) Lagrange's formula (ii) Newton's divided difference formula.

i) Lagranges formula

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)}(150) + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)}.392$$

$$+\frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)}(1452) + \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)}(2366)$$

$$+\frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)}(5202) = 810$$

$$f(9) = 810$$

ii)

5	150				
		121 $\frac{(392-150)}{(7-5)}$			
7	392		24		
		265		1	
11	1452		32		0

		457		1	
13	2366		42		
		709			
17	5202				

$f(9) = 150 + 121\ (9 - 5) + 24\ (9 - 5)\ (9 - 7) + 1(9 - 5)\ (9 - 7)\ (9 - 11) = 810$

3) **Using i) Langranges interpolation and ii) divided difference formula. Find the value of y when x = 10.**

x :	**5**	**6**	**9**	**11**
y :	**12**	**13**	**14**	**16**

i) Lagranges formula

$$y = f(10) = \frac{(10-6)\,(10-9)\,(10-11)}{(5-6)\,(5-9)\,(5-11)} \times 12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \times 13$$

$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \times 16$$

$$= \frac{44}{3}$$

ii)Divided difference

x	y	Δ	Δ^2	Δ^3
5	12			
		1		
6	13		$-\frac{2/3}{4}=\frac{-1}{6}$	
		$\frac{1}{3}$		$\frac{\frac{2}{15}+\frac{1}{6}}{11-5}=\frac{\frac{27}{90}}{6}=\frac{3/10}{6}=\frac{1}{20}$
9	14		$\frac{2/3}{5}=\frac{2}{15}$	
		$\frac{2}{2}=1$		
11	16			

$$f(10)=12+(10-5)+(10-5)(10-6).\left(-\frac{1}{6}\right)+(10-5)(10-6)(10-9)\left(\frac{1}{20}\right)$$

$$=\frac{44}{3}$$

4) **If y(1) = -3, y(3) = 9, y(4) =30, y(6) = 132 find the lagranges interpolating polynomial that takes the same values as y at the given points.**

Given:

x	1	3	4	6
y	-3	9	30	132

$$f(x) = \frac{(x-3)\,(x-4)\,(x-6)}{(1-3)(1-4)(1-6)} \cdot (-3) + \frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)} .9$$

$$+ \frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)} .30 + \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)} .132$$

$$= x^3 - 3x^2 + 5x - 6$$

5) Find the interpolating polynomial using Newton divided difference formula for the following data:

x	**0**	**1**	**2**	**5**
y	**2**	**3**	**12**	**147**

x	y	Δ	Δ^2	Δ^3
0	2			
		1		
1	3		4	
		9		1
2	12		9	
		45		
5	147			

$$F(x) = 2 + (x - 0)(1) + (x - 0)(x - 1)(4) + (x - 0)(x - 1)(x - 2)\,1$$

$$= x^3 + x^2 - x + 2$$

Introduction:

Limitations of analytical methods led to the evolution of Numerical methods. Numerical Methods often are repetitive in nature i.e., these consist of the repeated execution of the same procedure where at each step the result of the proceeding step is used. This process known as iterative process is continued until a desired degree of accuracy of the result is obtained.

Solution of Algebraic and Transcendental Equations:

The equation $f(x) = 0$ said to be purely algebraic if $f(x)$ is purely a polynomial in x.
If $f(x)$ contains some other functions like Trigonometric, Logarithmic, exponential etc. then $f(x) = 0$ is called a Transcendental equation.

Ex: (1) $x^4 - 7x^3 + 3x + 5 = 0$ is algebraic
(2) $e^x - x \tan x = 0$ is transcendental

Method of false position or Regula-Falsi Method:

This is another method to find the roots of $f(x) = 0$. This method is also known as Regular False Method.

In this method, we choose two points a and b such that $f(a)$ and $f(b)$ are of opposite signs.

$$x = x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

x_1 is the first approximation to the root of $f(x) = 0$.

EXAMPLE : 1

Find a real root of the equation $f(x) = x^3 - 2x - 5 = 0$ by method of False position.

SOLUTION

$$f(2) = -1 \text{ and } f(3) = 16$$

x	0	1	2	3
f(x)	-	-	-	+

Hence the root lies in between 2 and 3.

Take $a = 2, b = 3$.

$$\therefore \quad x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$= \frac{2(16) - 3(-1)}{16 - (-1)} = \frac{35}{17} = 2.058823529.$$

$$f(x_1) = f(2.058823529) = -0.390799917 < 0.$$

Therefore the root lies between 0.058823529 and 3. Again, using the formula, we get the second approximation as,

$$x_2 = \frac{2.058823529(16) - 3(-0.390799917)}{16 - (-0.390799917)} = 2.08126366$$

Proceeding like this, we get the next approximation as,

$$x_3 = 2.089639211,$$

$$x_4 = 2.092739575,$$

$$x_5 = 2.09388371,$$

$$x_6 = 2.094305452,$$

$$x_7 = 2.094460846$$

EXAMPLE 4.2

Determine the root of the equation $\cos x - x\,e^{x} = 0$ by the method of False position.

SOLUTION

x	0	1
f(x)	+	-

$$f(0) = 1 \text{ and } f(1) = -2.177979523$$

$\therefore$ $a = 0$ and $b = 1$. The root lies in between 0 and 1

$\therefore$
$$x_1 = \frac{0(-2.177979523) - 1(1)}{-2.177979523 - 1} = 0.3146653378$$

$$f(x_1) = f(0.314653378) = 0.51986.$$

$\therefore$ The root lies in between 0.314653378 and 1.

Hence,
$$x_2 = \frac{0.3146653378(-2.177979523) - 1(0.51986)}{-2.177979523 - 0.51986} = 0.44673$$

Proceeding like this, we get

$x_3 = 0.49402,$

$x_4 = 0.50995,$

$x_5 = 0.51520,$

$x_6 = 0.51692,$

EXAMPLE : 3

Determine the smallest positive root of $x - e^{-x} = 0$, correct of three significant figures using Regula False method.

SOLUTION

x	0	1
f(x)	-	+

Here, $f(0) = 0 - e^{-0} = -1$

and $f(1) = 1 - e^{-1} = 0.63212.$

$\therefore$ The smallest positive root lies in between 0 and 1. Here a = 0 and b = 1

$$\therefore \quad x_1 = \frac{0(0.63212) - 1(-1)}{0.63212 + 1} = 0.6127$$

$$f(0.6127) = 0.6127 - e^{-(0.6127)} = 0.0708$$

Hence, the next approximation lies in between 0 and 0.6127. Proceeding like this, we get

$x_2 = 0.57219{,}0 \qquad x_3 = 0.5677, \qquad x_4 = 0.5672, \qquad x_5 = 0.5671,$

Hence, the smallest positive root, which is correct up to three decimal places is,

$x = 0.567$

EXAMPLE : 4

Find the root of the equation $2x - \log x = 7$ which lies between 3.5 and 4 by Regula–False method.

Solution

Given $f(x) = 2x - \log x_{10} = 7$

Take $x_0 = 3.5, \quad x_1 = 4$

Using Regula Falsi method

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x)} . f(x_0)$$

$$x_2 = 3.5 - \frac{4 - 3.5}{(0.3979 + 0.5441)} (-0.5441)$$

$$x_2 = 3.7888$$

Now taking $x_0 = 3.7888$ and $x_1 = 4$

$$x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} . f(x_0)$$

$$x_3 = 3.7888 - \frac{4 - 3.7888}{0.3988} (-0.0009)$$

$$x_3 = 3.7893$$

The required root is = 3.789

EXAMPLE : 5

Find a real root of $xe^x = 3$ using Regula-Falsi method.

Solution

Given $f(x) = x\,e^x - 3 = 0$

$f(1) = e - 3 = -0.2817 < 0$

$f(2) = 2e^2 - 3 = 11.778 > 0$

$\therefore$ One root lies between 1 and 2

Now taking $x_0 = 1, x_1 = 2$

Using Regula – Falsi method

x	0	1	2
f(x)	-	-	+

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{1(11.778) - 2(-0.2817)}{11.778 + 0.2817}$$

$$x_2 = 1.329$$

Now $f(x_2) = f(1.329) = 1.329\ e^{1.329} - 3 = 2.0199 > 0$

$f(1) = -0.2817 < 0$

$\therefore$ The root lies between 1 and 1.329 taking $x_0 = 1$ and $x_2 = 1.329$

$\therefore$ Taking $x_0 = 1$ and $x_2 = 1.329$

$$\therefore \quad x_3 = \frac{x_0 f(x_2) - x_2 f(x_0)}{f(x_2) - f(x_0)}$$

$$= \frac{1(2.0199) + (1.329)(0.2817)}{(2.0199) + (0.2817)}$$

$$= \frac{2.3942}{2.3016} = 1.04$$

Now $f(x^3) = 1.04\ e^{1.04} - 3 = -0.05 < 0$

The root lies between x^2 and x^3

i.e., 1.04 and 1.329 $\quad [\because\ f(x_2) > 0 \text{ and } f(x_3) < 0]$

$$\therefore \quad x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = \frac{(1.04)(-0.05) - (1.329)(2.0199)}{(-0.05) - (2.0199)}$$

$x_4 = 1.08$ is the approximate root

EXAMPLE : 6

Find a real root of $e^x \sin x = 1$ using Regula – Falsi method

Solution

Given $f(x) = e^x \sin x - 1 = 0$

Consider $x_0 = 2$

$f(x_0) = f(2) = e^2 \sin 2 - 1 = -0.7421 < 0$

$f(x_1) = f(3) = e^3 \sin 3 - 1 = 0.511 > 0$

$\therefore$ The root lies between 2 and 3

Using Regula – Falsi method

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

x	0	1	2	3
f(x)	-	-	-	+

$$x_2 = \frac{2(0.511)+3(0.7421)}{0.511+0.7421}$$

$$x_2 = 2.93557$$

$$f(x_2) = e^{2.93557}\sin(2.93557) - 1$$

$$f(x_2) = -0.35538 < 0$$

$\therefore$ Root lies between x_2 and x_1

i.e., lies between 2.93557 and 3

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

$$= \frac{(2.93557)(0.511) - 3(-35538)}{0.511+0.35538}$$

$$x_3 = 2.96199$$

$$f(x_3) = e^{2.90199}\sin(2.96199) - 1 = -0.000819 < 0$$

$\therefore$ root lies between x_3 and x_1

$$x_4 = \frac{x_3 f(x_1) - x_1 f(x_3)}{f(x_1) - f(x_3)}$$

$$x_4 = \frac{2.96199(0.511)+3(0.000819)}{0.511+0.000819} = 2.9625898$$

$$f(x^4) = e^{2.9625898} \sin(2.9625898) - 1$$

$$f(x^4) = -0.0001898 < 0$$

$\therefore$ The root lies between x_4 and x_1

$$x_5 = \frac{x_4 f(x_1) - x_1 f(x_4)}{f(x_1) - f(x_4)}$$

$$= \frac{2.9625898(0.511) + 3(0.0001898)}{0.511 + (0.0001898)}$$

$$x_5 = 2.9626$$

we have

$$x_4 = 2.9625$$

$$x_5 = 2.9626$$

$\therefore$ $x_5 = x_4 = 2.962$

$\therefore$ The root lies between 2 and 3 is 2.962

Newton Raphson Method

This method is used to find the isolated roots of an equation f(x) = 0, when the derivative of f(x) is a simple expression.

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ is the first approximation

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ is the second approximation

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$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$ is the iterative formula for Newton Raphson Method

EXAMPLE : 1

1. **Using Newton's Raphson Method find the real root of $x \log_{10} x = 1.2$ correct to four decimal places.**

Answer:

x	0	1	2	3
f(x)	-	-	-	+

Let $f(x) = x \log_{10} x - 1.2$

$f(1) = -1.2$, $f(2) = -0.59794$, $f(3) = 0.23136$

We have $f(x) = \dfrac{x \log_e x}{\log_e 10} - 1.2 \Rightarrow f'(x) = \dfrac{1 + \log_e x}{\log_e 10}$

$= \log_{10} e + \log_{10} x$

$$\therefore \quad x_{k+1} = x_k - \frac{x_k \log_{10} x_k - 1.2}{\log_{10} e + \log_{10} x_K}$$

Let $x_0 = 2.5$ (you may choose 2 or 3 also)

$$x_1 = 2.5 - \frac{2.5 \log_{10} 2.5 - 1.2}{\log_{10} e + \log_{10} 2.5} = 2.7465$$

$$x_2 = 2.7465 - \frac{2.7465 \log 2.7465 - 1.2}{\log_{10} e + \log_{10} 2.7465} = 2.7406$$

Repeating the procedure

$x_3 = 2.7406$

$\therefore \quad x \approx 2.7406$ is the root of the given equation

EXAMPLE : 2

2.Using Newton's Method, find the real root of $xe^x = 2$. Correct to 3 decimal places.

Answer:

Let $f(x) = xe^x - 2$

$f(0) = -2$

$f(1) = e - 2 = 0.7182$

x	0	1
f(x)	-	+

Let $x_0 = 1$

$f'(x) = (x + 1)\, e^x$

We have

$$x_{k+1} = x_k - \frac{x_k e^{x_k} - 2}{(x_k + 1)\, e^{x_k}}$$

$$x_1 = 1 - \frac{e - 2}{2e} = 0.8678$$

$$x_2 = 0.8678 - \frac{(0.8678)\, e^{0.8678} - 2}{(1.8678)\, e^{0.8678}} = 0.8527$$

$$x_3 = 0.8527 - \frac{(0.8527)\, e^{0.8527} - 2}{(1.8527)\, e^{0.8527}} = 0.8526$$

$\therefore$ $x \approx 0.8526$, is the required root. Correct to 3 decimal places

EXAMPLE : 3

3. Find by Newton's Method the real root of 3x = cosx + 1 near 0.6, x is in radians. Correct for four decimal places.

Answer:

Let $f(x) = 3x - \cos x - 1$

$f'(x) = 3 + \sin x$

$$x_{k+1} = x_k - \frac{3x_k - \cos x_{k-1}}{3 + \sin x_k}$$

When $x_0 = 0.6$ $\quad x_1 = 0.6 - \dfrac{3(0.6) - \cos(0.6) - 1}{3 + \sin(0.6)} = 0.6071$

$$x_2 = 0.6071 - \frac{3(0.6071) - \cos(0.6071) - 1}{3 + \sin(0.6071)} = 0.6071$$

Since $x_1 = x_2$

The desired root is 0.6071

EXAMPLE : 4

Find a root of $x \sin x + \cos x = 0$, using Newton – Raphson method near x=π

SOLUTION

$$f(x) = x \sin x + \cos x.$$

$$\therefore \quad f'(x) = \sin x + x \cos x - \sin x = x \cos x$$

The Newton – Raphson method formula is,

$$x_{n+1} = x_n - \frac{x_n \sin x_n + \cos x_n}{x_n \cos x_n}, \qquad n = 0, 1, 2, \ldots.$$

Let $x_0 = \pi = 3.1416.$

$$\therefore \quad x_1 = 3.1416 - \frac{3.1416 \sin \pi + \cos \pi}{3.1416 \cos \pi} = 2.8233.$$

Similarly,

$x_2 = 2.7986$

$x_3 = 2.7984$

$x_4 = 2.7984$

$\therefore$ $x = 2.7984$ can be taken as a root of the equation $x \sin x + \cos x = 0$.

EXAMPLE : 5

Find the smallest positive root of $x - e^{-x} = 0$, using Newton – Raphson method.

SOLUTION

Here $f(x) = x - e^{-x}$

$f^1(x) = 1 + e^{-x}$

$f(0) = -1$ and $f(1) = 0.63212$.

x	0	1
f(x)	-	+

$\therefore$ The smallest positive root of $f(x) = 0$ lies in between 0 and 1.

Let $x_0 = 1$

The Newton – Raphson method formula is,

$$x_{n+1} = x_n - \frac{x_n - e^{-x}n}{1 + e^{-x}n}, \; n = 0, 1, 2, \ldots\ldots$$

$f(0) = f(1) = 0.63212$

$f'(0) = f'(1) = 1.3679$

$$\therefore \quad x_1 = x_0 - \frac{x_0 - e^{-x_0}}{1 + e^{-x_0}} = 1 - \frac{0.63212}{1.3679} = 0.5379.$$

$f(0.5379) = -0.0461$

$f'(0.5379) = 1.584.$

$$\therefore \quad x_2 = 0.5379 + \frac{0.0461}{1.584} = 0.567$$

Similarly, $x_3 = 0.56714$

$\therefore$ $x = 0.567$ can be taken as the smallest positive root of $x - e^{-x} = 0.$, correct to three decimal places.

6 **Obtain the iterative formula for finding the square root of N and find $\sqrt{41}$**

Answer:

Let $x = \sqrt{N}$

or $x^2 - N = 0$

$\therefore f(x) = x^2 - N$

$f'(x) = 2x$

Now

$$x_{k+1} = x_k - \frac{x_k^2 - N}{2x_k}$$

$$= x_k - \frac{x_k}{2} + \frac{N}{2x_k}$$

i.e. $$x_{k+1} = \frac{1}{2}\left\{x_k + \frac{N}{x_k}\right\}$$

To find $\sqrt{41}$

Observe that $\sqrt{36} < \sqrt{41}$

$\therefore$ Choose $x_0 = 6$

$$x_1 = \frac{1}{2}\left\{6 + \frac{41}{6}\right\} = 6.4166$$

$$x_2 = \frac{1}{2}\left\{6.4166 + \frac{41}{6.4166}\right\} = 6.4031$$

$$x_3 = \frac{1}{2}\left\{6.4031 + \frac{41}{6.4031}\right\} = 6.4031$$

Since $x_2 = x_3 = 6.4031$

The value of $\sqrt{41} \approx 6.4031$

7. **Obtain an iterative formula for finding the p-th root of N and hence find $(10)^{1/3}$ correct to 3 decimal places.**

Answer:

Let $x^p = N$

or $x^p - N = 0$

Let $f(x) = x^p - N$

$f'(x) = px^{p-1}$

Now $x_{k+1} = x_k - \dfrac{x_k^p - N}{px_k^{p-1}}$

Observe that $8 < 10$

$\Rightarrow \quad 8^{1/3} < 10^{1/3}$

i.e. $\quad 2 < (10)^{1/3}$

$\therefore \quad$ Use $x_0 = 2,\ p = 3,\ N=10$

$$x_1 = 2 - \frac{2^3 - 10}{3(2^2)} = 2.1666$$

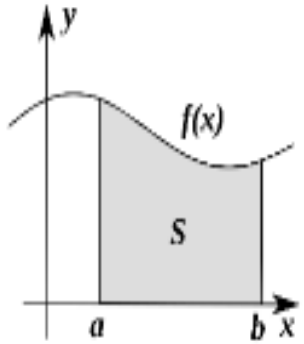
$$x_2 = 2.1666 - \frac{(2.1666)^3 - 10}{3(2.1666)^2} = 2.1545$$

$$x_3 = 2.1545 - \frac{(2.1545)^3 - 10}{3(2.1545)^2} = 2.1544$$

$\therefore \quad (10)^{1/3} \approx 2.1544$

Numerical Integration

In numerical integration, we compute an approximate solution to a definite integral $\int_a^b f(x)\,dx$ which gives the area under the curve $f(x)$.



Divided the interval $[a\,,b]$ into n equal parts such that $h = \frac{b-a}{n}$. Then prepare the following table

$a = x_0$	$x_1 = x_0 + h$	$x_2 = x_1 + h$	....	$x_{n-1} = x_{n-2} + h$	$x_n = b$
$y_0 = f(x_0)$	$y_1 = f(x_1)$	$y_2 = f(x_2)$	....	$y_{n-1} = f(x_{n-1})$	$y_n = f(x_n)$

1. **Simpon's (1/3)rd rule: (n should be multiple of 2)**

$$\int_a^b y\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \cdots + y_{n-2})]$$

2. **Simpon's (3/8)th rule: (n should be multiple of 3)**

$$\int_a^b y\,dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \cdots + y_{n-3})]$$

3. **Weddle's rule: (n should be multiple of 6)**

$$\int_a^b y\,dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6] \text{ (for } n = 6)$$

Remember: Coefficient of y_i s are (1,5,1,6,1,5,1)

$$\int_a^b y\,dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}]$$
(for $n = 12$)

3) **Using Simpson's $\frac{1}{3}^{rd}$ rule evaluate $\int_0^1 \frac{dx}{1+x^2}$ by dividing the interval $(0\ ,\ 1)$ into 4 equal sub intervals and hence find the value of π correct to four decimal places.**

Solution: Let us divide [0,1] into 4 equal strips (n = 4)

$\therefore$ length of each strip: $h = \frac{1-0}{4} = \frac{1}{4}$

The points of division are $x = 0, \frac{1}{4}, \frac{2}{4} = \frac{1}{2}, \frac{3}{4}, \frac{4}{4} = 1$

By data $y = \frac{1}{1+x^2}$

Now we have the following table

x	0	1/4	1/2	3/4	1
$y = \frac{1}{1+x^2}$	1	16/17	4/5	16/25	1/2
	y_0	y_1	y_2	y_3	y_4

$$I = \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$I = 0.7854$$

To deduce the value of π: We perform theoretical integration and equate the resulting value to the numerical value obtained.

$$\therefore \int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1} x\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}$$

$$We\,must\,have,\ \frac{\pi}{4} = 0.7854 \Rightarrow \pi = 4(0.7854) = 3.1416$$

$$Thus\ \ \boxed{\pi = 3.1416}$$

4) **Given that**

x	4	4.2	4.4	4.6	4.8	5	5.2
$\log x$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

Evaluate $\int_4^{5.2} \log x\ dx$ **using Simpson's** $\frac{3}{8}^{th}$ **rule**

Solution: Simpson's $\frac{3}{8}^{th}$ rule for n = 6 is given by

$$\int_a^b y dx = \frac{3h}{8}\left[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)\right]$$

$$\int_4^{5.2} \log_e x dx = \frac{3(0.2)}{8}\left[(1.3863 + 1.6487) + 3(1.4351 + 1.4816 + 1.5686 + 1.6094) + 2(1.5261)\right]$$

$$\int_4^{5.2} \log_e x dx = 1.8279$$

5) **Using Weddle's rule evaluate** $\int_0^1 \frac{xdx}{1+x^2}$ **by taking seven ordinates and hence find** $\log_e 2$

Solution: Let us divide [0,1] into 6 equal strips (since seven ordinates)

$\therefore$ length of each strip: $h = \frac{1-0}{6} = \frac{1}{6}$

The points of division are $x = 0, \frac{1}{6}, \frac{2}{6} = \frac{1}{3}, \frac{3}{6} = \frac{1}{2}, \frac{4}{6} = \frac{2}{3}, \frac{5}{6}, \frac{6}{6} = 1$

By data $y = \frac{1}{1+x^2}$

Now we have the following table

x	0	1/6	1/3	1/2	2/3	5/6	1
$y = \frac{x}{1+x^2}$	0	6/37	3/10	2/5	6/13	30/61	1/2
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Weddle's rule for n = 6 is given by

$$\int_a^b y dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

$$\int_0^1 \frac{x}{1+x^2} dx = \frac{3(1/6)}{10}\left[0 + 5(6/37) + 3/10 + 6(2/5) + 6/13 + 5(30/61) + 1/2\right]$$

$$\int_0^1 \frac{x}{1+x^2} dx = 0.3466$$

To deduce the value of $\log_e 2$: We perform theoretical integration and equate the resulting value to the numerical value obtained.

$$\therefore \int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\log_e(1+x^2)\Big]_0^1 = \frac{1}{2}\log_e 2 - \frac{1}{2}\log_e 1$$

$$\textit{Hence} \int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\log_e 2$$

$$\textit{We must have,}\ \frac{1}{2}\log_e 2 = 0.3466 \Rightarrow \log_e 2 = 2(0.3466) = 0.6932$$

$$\textit{Thus}\ \boxed{\log_e 2 = 0.6932}$$